Stratifications for group actions & moduli problems

Talk 4: Stratifications for moduli of quiver representations
by V. Hoskins

§1 Moduli of quiver representations

Def^n A quiver $Q=(V,A,h,t)$ is a set of vertices V and a set of arrows A with head and tail maps $h,t: A \to V$.

Eg $\bullet\circlearrowleft$ Jordan quiver, $\bullet \rightrightarrows \bullet$ Kronecker quiver etc.

A representation of Q in an abelian category $\mathcal{A}$ is $(C_v, v\in V; f_a, a\in A)$ where $C_v \in \operatorname{ob}\mathcal{A}$ and $f_a : C_{t(a)} \to C_{h(a)}$ is a morphism in $\mathcal{A}$.

Main example: $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$ the category of $\mathbb{C}$-vector spaces. We will refer to representations of Q in $\mathrm{Vect}_{\mathbb{C}}$ just as representations of Q.

A family of representations of Q over a variety S is a representation of Q in the category of locally free sheaves on S: $\mathcal{F}=(\mathcal{E}_v, v\in V, f_a: \mathcal{E}_{t(a)} \to \mathcal{E}_{h(a)}, a\in A)$.
For each $s\in S$, $\mathcal{F}_s$ is a representation of Q (in $\mathrm{Vect}_{\mathbb{C}}$).

Def^n: A morphism of representations of Q

$$\Phi : W=(W_v, v\in V, f_a, a\in A) \longrightarrow W'=(W'_v, v\in V, f'_a, a\in A)$$

is given by morphisms $\Phi_v : W_v \to W'_v$ such that $\forall\, a\in A$, we have a commutative square

$$\begin{array}{ccc} W_{t(a)} & \xrightarrow{f_a} & W_{h(a)} \\ \downarrow{\scriptstyle \Phi_{t(a)}} & & \downarrow{\scriptstyle \Phi_{h(a)}} \\ W'_{t(a)} & \xrightarrow{f'_a} & W'_{h(a)}. \end{array}$$

The dimension vector of a representation of Q is $\underline{d}(W) = (\dim W_v)_{v\in V} \in \mathbb{N}^V$.

Moduli problem: Classify representations of Q (in $\mathrm{Vect}_{\mathbb{C}}$) of fixed dimension vector $\underline{d} \in \mathbb{N}^V$ up to isomorphism.

Proposition: Over the affine space

$$R = \mathrm{Rep}_{\underline{d}} Q = \bigoplus_{a\in A} \mathrm{Hom}(\mathbb{C}^{t(a)}, \mathbb{C}^{h(a)})$$

there is a tautological family $\mathcal{T}$ of representations of Q of dim $\underline{d}$ with the local universal property. (see below)

Furthermore, the group $GL_{\underline{d}} = \prod_{v\in V} GL_{d_v}$ acts on R by

$$GL_{\underline{d}} \times \mathrm{Rep}_{\underline{d}} Q \longrightarrow \mathrm{Rep}_{\underline{d}} Q$$

$$((g_v)_{v\in V}, (f_a)_{a\in A}) \longmapsto (g_{h(a)} \circ f_a \circ g_{t(a)}^{-1})_{a\in A}$$

such that there is a bijective correspondence:

$$\{GL_{\underline{d}}\text{-orbits in } \mathrm{Rep}_{\underline{d}} Q\} \leftrightarrow \left\{\begin{array}{l}\text{iso classes of } \underline{d}\text{-dim}^{e} \\ \text{representations of } Q\end{array}\right\}.$$

In particular, any coarse moduli space of $\underline{d}$-dim^e reps of Q is a categorical quotient of $GL_{\underline{d}} \curvearrowright \mathrm{Rep}_{\underline{d}} Q$.

Proof: $\mathcal{T} := (\mathcal{O}_R^{\oplus d_v}, v\in V;\ f_a : \mathcal{O}_R^{\oplus d_{t(a)}} \longrightarrow \mathcal{O}_R^{\oplus d_{h(a)}}, a\in A)$ is the tautological family. ↑ over $r = (r_a)_{a\in A} \in R$ this is the morphism r_a.

By def^n, $\mathcal{T}$ has the local universal property if for every family $\mathcal{F} = (\mathcal{E}_v, \ell_a)$ over a variety S and $\forall\, s\in S$, $\exists\, s\in U \subseteq S$ open and $\phi : U \to R$ such that $\phi^* \mathcal{T} \cong \mathcal{F}|_U$.

We define U to be an open set containing s on which each locally free sheaf $\mathcal{E}_v$ is trivialisable $\psi_v : \mathcal{E}_v|_U \xrightarrow{\simeq} \mathcal{O}_U^{\oplus d_v}$, then define $\phi = (\phi_a)_{a\in A}$ by the homomorphisms

$$\phi_a : \mathcal{O}_U^{\oplus d_{t(a)}} \xrightarrow[\psi_{t(a)}]{} \mathcal{E}_{t(a)}|_U \xrightarrow{\ell_a} \mathcal{E}_{h(a)}|_U \xrightarrow[\psi_{h(a)}]{} \mathcal{O}_U^{\oplus d_{h(a)}}.$$

In particular, if $S = \mathrm{Spec}\, k$, we see any rep W of Q is isomorphic to a representation $\mathcal{T}_r$ for some $r\in R$.

Furthermore, $r, r' \in R$ are iso reps of Q $\iff$ $\exists\, g_v : \mathbb{C}^{d_v} \xrightarrow{\simeq} \mathbb{C}^{d_v}$ commuting with the arrows

i.e. $GL_{\underline{d}} \cdot r = GL_{\underline{d}} \cdot r'$. ☒

The affine GIT quotient

$$\mathrm{Rep}_{\underline{d}}\, Q \longrightarrow \mathrm{Rep}_{\underline{d}}\, Q /\!\!/ GL_{\underline{d}} := \mathrm{Spec}\ \mathcal{O}(\mathrm{Rep}_{\underline{d}} Q)^{GL_{\underline{d}}}$$

is a categorical quotient which parametrises the closed $GL_{\underline{d}}$-orbits in $\mathrm{Rep}_{\underline{d}}\, Q$.

Ex $Q = \bullet \longrightarrow \bullet$ $\quad \underline{d} = (n,m)$ $\quad GL_{\underline{d}} = GL_n \times GL_m \curvearrowright R = \mathrm{Mat}_{m\times n}$

The orbits are matrices of a fixed rank & the only closed orbit is the zero matrix $\Rightarrow R /\!\!/ GL_{\underline{d}} = \mathrm{Spec}\, k = *$.

Theorem (Le Bruyn - Procesi)

$\mathcal{O}(\mathrm{Rep}_{\underline{d}}\, Q)^{GL_{\underline{d}}}$ is generated by traces of oriented cycles in Q. Hence, for Q acyclic, $\mathrm{Rep}_{\underline{d}}\, Q /\!\!/ GL_{\underline{d}} = *$.

§2 Affine GIT using a linearisation by a character

Often for G reductive $\curvearrowright R = \mathbb{A}^n$, we have $R /\!\!/ G = *$.

Solution: Throw out "bad" orbits (eg. $0 \in R$) by "linearising" the action to obtain a non-trivial notion of semistability.

Def^n: For a character $\chi : G \to \mathbb{G}_m$, define a G-action on $\mathrm{Tot}(\mathcal{O}_R) = R \times \mathbb{A}^1$ by

$$G \times \mathrm{Tot}(\mathcal{O}_R) \longrightarrow \mathrm{Tot}(\mathcal{O}_R)$$
$$g \cdot (r, z) = (g \cdot r, \chi(g) z)$$

Notation: $\mathcal{O}_R^\chi$ denotes the line bundle $\mathcal{O}_R$ with this given G-action. This is a linearisation of $G \curvearrowright R$.

Use invariant sections of $\mathcal{O}_R^\chi$ to construct a GIT quotient:

$$H^0(R, \mathcal{O}_R^\chi)^G = \{ f \in \mathcal{O}(R) : f(g\cdot r) = \chi(g) f(r) \ \forall g \in G, r \in R \} =: \mathcal{O}(R)^{G,\chi}$$

$\uparrow$ χ-semi-invariant functions

Pf:

$$\mathcal{O}(R) \cong H^0(R, \mathcal{O}_R^\chi)^G$$
$$f \mapsto \sigma_f(r) = (r, f(r))$$

Then $(g \cdot \sigma_f)(r) = \sigma_f(r) \iff f(r) = \chi(g) f(g^{-1}\cdot r)$.

$$(g\cdot\sigma_f)(r) = (r, \chi(g) f(g^{-1}\cdot r))$$

The inclusion $\bigoplus_{n\geq 0} H^0(R, \mathcal{O}_R^{\chi^n})^G \hookrightarrow \bigoplus_{n \geq 0} H^0(R, \mathcal{O}_R^{\chi})$

induces a rational map

$$R \overset{\varphi}{- - \to} R/\!/_{\chi} G := \operatorname{Proj} \bigoplus_{n\geq 0} H^0(R, \mathcal{O}_R^{\chi^n})^G \xrightarrow[\text{proj}]{} R/\!/ G = \operatorname{Spec} H^0(R, \mathcal{O})^G$$

open ∪I

$$R^{\chi\text{-ss}} = \{ r \in R : \exists f \in \mathcal{O}(R)^{G,\chi^n} \text{ for } n > 0 \text{ s.t. } f(r) \neq 0 \}$$

domain of definition of φ is the χ-semistable locus.

Hilbert-Mumford Criterion

Let G be a reductive gp acting linearly on an affine space R and linearise the action using $\chi : G \to \mathbb{G}_m$.
Then $r \in R$ is χ-semistable $\iff \mu^{\chi}(r, \lambda) \geq 0 \;\forall$ 1-PSs $\lambda : \mathbb{G}_m \to G$ s.t. $\lim_{t \to 0} \lambda(t) \cdot r$ exists

where $\mu^{\chi}(r, \lambda) := \langle \chi, \lambda \rangle$ is the unique $m \in \mathbb{Z}$ s.t. $\chi \circ \lambda(t) = t^m$.

§3 King's Construction of moduli spaces of quiver reps

For $GL_{\underline{d}} \curvearrowright \operatorname{Rep}_{\underline{d}} Q$ linearise the action using a character.

$$\underline{\theta} = (\theta_v)_{v \in V} \in \mathbb{Z}^V \leftrightarrow \chi_{\underline{\theta}} : GL_{\underline{d}} \longrightarrow \mathbb{G}_m \quad \text{character}$$
$$(g_v)_{v\in V} \mapsto \prod_{v \in V} \det g_v^{\theta_v}$$

There is $\Delta \hookrightarrow G = GL_{\underline{d}}$ such that $\Delta \subseteq G_r \;\forall r \in R$.

$\Delta \cong \mathbb{G}_m \ni t \mapsto (t I_{d_v})_{v \in V}$ global stabiliser (of Δ)

For $V^{\chi_\theta\text{-ss}}$ to be nonempty, we need $\chi(\Delta) = 1 \left(\iff \sum_{v \in V} \theta_v d_v = 0.\right)$ ← Assume

Def: A $\underline{d}$-dim^e representation W of Q is $\underline{\theta}$-semistable if $\forall$ subrepresentations $W' \subseteq W$, $\underline{\theta}(W') = \sum_{v \in V} \theta_v \dim W'_v \geq 0$.

Thm (King)
$\operatorname{Rep}_{\underline{d}} Q /\!/_{\chi_\theta} GL_{\underline{d}}$ is a (coarse) moduli space for (S-equiv. classes of) θ-semistable $\underline{d}$-dimensional representations of Q.

In particular, GIT semistability for χ_θ $\iff$ θ-semistability for reps of Q.

§4 Hesselink & Morse Stratifications of affine spaces

For a cx reductive gp $G = K_{\mathbb{C}}$ acting linearly on a cx affine space R with respect to $\mathcal{O}_R^{\chi}$, for a character $\chi : G \to \mathbb{G}_m$, one can also construct using a K-inv. norm $\|-\|$ on $\mathfrak{k}$:

(1) A Hesselink stratification

$$R = \bigsqcup_{[\lambda]} S^H_{[\lambda]} \quad \text{where} \quad S^H_{[\lambda]} := \{ r \in R : \exists\, \lambda \in [\lambda] \text{ that is } \chi\text{-adapted to } r \}$$

Def^n: A 1-PS λ is χ-adapted to $r \in R^{\chi\text{-us}} = R - R^{\chi\text{-ss}}$ if $\lim_{t \to 0} \lambda(t) \cdot r$ exists and

$$\frac{\mu^{\chi}(r, \lambda)}{\|\lambda\|} = \frac{\langle \chi, \lambda \rangle}{\|\lambda\|} = M^{\chi}(r) := \inf_{\substack{\lambda' \text{ s.t. } \lim_{t\to 0} \lambda'(t)\cdot r \\ \text{exists}}} \frac{\langle \chi, \lambda' \rangle}{\|\lambda'\|}.$$

(the additional index d is redundant, as $d = \frac{\langle \chi, \lambda \rangle}{\|\lambda\|}$.)

(2) A Morse stratification for $\|\mu_\chi\|^2 : R \to \mathbb{R}$

where $\mu_\chi : R \to \mathfrak{k}^*$ (moment map) for $K \subseteq G$ max^l compact

$$\mu_\chi(r) \cdot A := \tfrac{1}{2} \omega(A_r, r) - d\chi \cdot A \quad \text{for } d\chi : \mathfrak{k} \to \mathbb{R} \cong \text{Lie } S^1$$

$$\text{Crit} \|\mu_\chi\|^2 = \bigsqcup_{K\cdot\beta} C_{K\cdot\beta} \quad \text{where } C_{K\cdot\beta} = \text{Crit}\|\mu_\chi\|^2 \cap \mu_\chi^{-1}(K\cdot\beta)$$

($K\cdot\beta$: coadjoint orbits)

eg $C_0 = \mu_\chi^{-1}(0) \subseteq \text{Crit}\|\mu_\chi\|^2$ lowest critical locus.

$$R = \bigsqcup_{K\cdot\beta} S^M_{K\cdot\beta} \quad \text{where} \quad S^M_{K\cdot\beta} = \{ r \in R : \lim_{t\to\infty} \varphi_t(r) \in C_{K\cdot\beta} \}$$

[Harada-Wilkin]

(φ_t: negative gradient flow of r under $\|\mu_\chi\|^2$)

Theorem 1 (H.)

Let $G = K_{\mathbb{C}} \curvearrowright R = \mathbb{A}^n$. For any character $\chi : G \to \mathbb{G}_m$ and any K-invariant norm $\|-\|$ on $\mathfrak{k}$, the (GIT) Hesselink stratification and (symplectic) Morse stratification coincide.

The proof of this relies on:

<u>Theorem 2</u> "Affine Kempf-Ness w.r.t. a character"

(i) $\overline{G\cdot r} \cap \mu_\chi^{-1}(0) \neq \emptyset \iff r \in R^{\chi-ss}$

(ii) The lowest strata coincide: $S_0^M = S_0^H = R^{\chi-ss}$ and there is a homeomorphism $\mu_\chi^{-1}(0)/K \simeq R/\!/_\chi G$.

<u>Sketch proof of Thm 1</u>:

We show the Morse strata $S^M_{K\cdot\beta}$ agree with the "Kirwan strata" $G p_\beta^{-1}(Z_\beta)$ where $p_\beta : R^\beta_+ \to R^\beta = \mathrm{Crit}\,\mu_{\chi,\beta}$ & Z_β = lowest Morse stratum for $\|\mu_\chi - \beta\|^2$ on R^β.

(<u>recall</u> $\mu_{\chi,\beta} : R \to \mathbb{R}$, $r \mapsto \mu_\chi(r)\cdot\beta$ is a Morse-Bott function)

The Hesselink strata have a similar structure:

$$S^H_{[\lambda]} = G p_\lambda^{-1}(Z_\lambda) \quad \text{where} \quad p_\lambda : V^\lambda_+ \to V^\lambda$$

Z_λ = GIT semistable set for action of smaller reductive group (the Levi in $P(\lambda)$) on V^λ w.r.t. modified linearisation. ⇝ alter character

By Thm 2, $Z_\beta = Z_{\lambda_\beta}$ where λ_β is a 1-PS associated to β

Hence $S^M_{K\cdot\beta} = S^H_{[\lambda_\beta]}$ as required. ∎

§5 <u>The Harder-Narasimhan Stratification on $\mathrm{Rep}_{\underline{d}} Q$</u>

<u>Idea</u>: a Harder-Narasimhan (HN) filtration is a "maximally destabilising filtration" (Semistable objects have trivial HN filtr.s).

⇝ Need notion of semistability for reps of Q of any dim.

Use stability parameters $\underline{\alpha} \in \mathbb{N}^V$ & $\underline{\theta} \in \mathbb{Z}^V$ s.t. $\sum_{v\in V} d_v\theta_v = 0$.

<u>Def^n</u>: A representation W of Q is <u>(θ,α)-semistable</u> if for all subrepresentations $0 \neq W' \subsetneq W$, $\dfrac{\underline{\theta}(W')}{\underline{\alpha}(W')} \geqslant \dfrac{\underline{\theta}(W)}{\underline{\alpha}(W)}$.

<u>Rmk</u>: As $\sum \theta_v d_v = 0$, this extends the notion of θ-semistability for $\underline{d}$-dim^e reps to arbitrary dim^e representations.

Lemma Every representation W of Q has a unique <u>Harder-Narasimhan filtration</u> w.r.t. $(\underline{\theta}, \underline{\alpha})$

$$0 = W^{(0)} \subseteq W^{(1)} \subseteq \dots \subseteq W^{(s)} = W$$

where $W^i = W^{(i)}/W^{(i-1)}$ are $(\underline{\theta}, \underline{\alpha})$-semistable and

$$\frac{\underline{\theta}(W^1)}{\underline{\alpha}(W^1)} < \dots\dots < \frac{\underline{\theta}(W^s)}{\underline{\alpha}(W^s)}.$$

The HN type of W is $\tau(W) = (\underline{\dim} W^1, \dots, \underline{\dim} W^s)$.

We can stratify $\mathrm{Rep}_{\underline{d}} Q$ by HN types to obtain a <u>Harder-Narasimhan Stratification</u>: $\mathrm{Rep}_{\underline{d}} Q = \coprod_\tau S_\tau^{HN}$ [Shatz; Reineke].

<u>Recall</u> (from talk 1): $\alpha \in \mathbb{N}^V$ defines a norm on 1-PSs of $GL_{\underline{d}} = \prod_{v \in V} GL_{d_v}$ by $\|(\lambda_v)_{v \in V}\|_\alpha^2 = \sum_{v \in V} \alpha_v \|\lambda_v\|^2$.
↑ Euclidean norm on GL.

<u>Theorem 3</u> (H.)

For a quiver $Q = (V, A, h, t)$, fix a dimension vector $\underline{d}$ and stability parameters $\theta \in \mathbb{Z}^V$ s.t $\sum_{v \in V} \theta_v d_v = 0$ & $\alpha \in \mathbb{N}^V$.
Then the following stratifications on $\mathrm{Rep}_{\underline{d}} Q$ coincide:

(1) The Harder-Narasimhan stratification $\mathrm{Rep}_{\underline{d}} Q = \coprod_\tau S_\tau^{HN}$ w.r.t. $(\underline{\theta}, \underline{\alpha})$.

(2) The Hesselink stratification for $GL_{\underline{d}} \curvearrowright \mathrm{Rep}_{\underline{d}} Q$ w.r.t. the linearisation $\chi_\theta : GL_{\underline{d}} \to \mathbb{G}_m$ and norm $\|-\|_\alpha$. $\mathrm{Rep}_{\underline{d}} Q = \coprod_{[\lambda]} S_{[\lambda]}^H$

(3) The Morse stratification for $\|\mu_{\chi_\theta}\|_\alpha^2 : \mathrm{Rep}_{\underline{d}} Q \to \mathbb{R}$
$\mathrm{Rep}_{\underline{d}} Q = \coprod_\beta S_{K \cdot \beta}^M$ where $K = \prod_{v \in V} U(d_v) \subseteq GL_{\underline{d}}$ & $\mu_\chi : \mathrm{Rep}_{\underline{d}} Q \to \mathfrak{k}^*$ moment map. (shifted by χ)

Outline of proof "(1) = (2)"

(a) Correspondence of indices:

$\tau = (\underline{d}^1, \dots, \underline{d}^s) \longleftrightarrow K \cdot \beta_\tau = K \cdot (\beta_{\tau,v})_{v \in V} \longleftrightarrow [\lambda_\tau] = [(\lambda_{\tau,v})]_{v \in V}$

HN type for Q w.r.t. $(\underline{\theta}, \underline{\alpha})$ — coadjoint orbit (Morse index) — conj. class of 1-PS of G (Hesselink index)

$\beta_{\tau,v} \in \mathfrak{t}_{v,+}$ where $T_v \subseteq U(d_v)$ diagonal torus.

$\tau \rightsquigarrow K \cdot \beta_\tau$:

$$\beta_{\tau,v} = 2\pi i \, \mathrm{diag}(\underbrace{\beta_1, \dots, \beta_1}_{d_v^1}, \beta_2, \dots, \beta_2, \dots, \underbrace{\beta_s, \dots \beta_s}_{d_v^s}) \in \mathfrak{t}_{v,+}$$

↗ diagonal entries are decreasing

where $\beta_i := -\dfrac{\theta(\underline{d}^i)}{\alpha(\underline{d}^i)} \in \mathbb{Q}$.

(b) King's work on semistable quiver representations

$\Rightarrow$ lowest strata agree.

(c) Show higher strata agree by proving limit sets coincide: $Z_\tau^{HN} = Z_{\lambda_\tau}^{H}$ using b),

where $S_\tau^{HN} = Gp_\tau^{-1}(Z_\tau^{HN})$ & $S_{\lambda_\tau}^{H} = Gp_{\lambda_\tau}^{-1}(Z_{\lambda_\tau}^{HN})$.

<u>Rmk</u>: Harada & Wilkin prove (1) $\Leftrightarrow$ (3).