Stratifications for group actions and moduli problems

Talk 3 : The Kempf-Ness Theorem

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§1 The Statement of the K-N Theorem

Let G be a reductive group$/\mathbb{C}$; then $G = K_{\mathbb{C}}$ for $K \subseteq G$ max.^{l} compact

e.g. $G = GL_n \supseteq K = U(n)$.

Suppose G acts linearly on a smooth projective variety$/\mathbb{C}$ $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ via $G \to GL_{n+1}$.

We can pick coords on $\mathbb{P}^n$ so that K acts unitarily i.e. via $K \to U(n+1)$.

The smooth proj. variety $X \overset{i}{\subseteq} \mathbb{P}^n$ is a symplectic mfld with symplectic form $\omega = i^* \omega_{FS}$ & the K-action is symplectic.

We have two quotients associated to this action:

(1) The projective GIT quotient $\pi : X^{ss} \longrightarrow X/\!/G$,

(2) The symplectic reduction (at 0) $p : \mu^{-1}(0) \to \mu^{-1}(0)/K$.

<u>Theorem</u> (Kempf-Ness) $G = K_{\mathbb{C}} \curvearrowright X \subseteq \mathbb{P}^n$ linearly.

Let $x \in X$; then

(i) $\overline{G\cdot x} \cap \mu^{-1}(0) \neq \emptyset \iff x \in X^{ss}$.

(ii) $G\cdot x \cap \mu^{-1}(0) \neq \emptyset \iff x \in X^{ps} := \{x \in X^{ss} : G\cdot x \subseteq X^{ss} \text{ closed}\}$.

In this case, $G\cdot x \cap \mu^{-1}(0)$ is a single K-orbit.

(iii) The inclusion $\mu^{-1}(0) \hookrightarrow X^{ss}$ induces a homeomorphism

$$\begin{array}{ccc} \mu^{-1}(0) & \hookrightarrow & X^{ss} \\ \downarrow & & \downarrow \\ \mu^{-1}(0)/K & \xrightarrow{\sim} & X/\!/G. \end{array}$$

(iv) 0 is a regular value of the moment map $\mu : X \to \mathfrak{k}^*$ $\iff X^s = X^{ss}$.

<u>Example</u> Let $G = \mathbb{C}^* \curvearrowright X = \mathbb{P}^n$ by $t\cdot[x_0 : \quad : x_n] = [t^{-1}x_0 : tx_1 \quad : tx_n]$

<u>Talk 1</u> : $X^{ss} = \left\{ [x_0 : \quad : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ \& } x_i \neq 0 \text{ for some } 1 \leq i \leq n \right\} \to X/\!/G = \mathbb{P}^{n-1}$

For $K = U(1) \curvearrowright X$, we have moment map $\mu : X \to \mathbb{R} \cong \mathfrak{u}(1)^*$

$$\mu([x_0 : \quad : x_n]) = \frac{1}{2}\left(-|x_0|^2 + |x_1|^2 + \dots + |x_n|^2\right) / \sum_{j=0}^{n} |x_j|^2.$$

Then $\mu^{-1}(0) = \{ [x_0 : \ : x_n] \in \mathbb{P}^n : \sum_{j=1}^{n} \frac{|x_j|^2}{|x_0|^2} = 1 \} = S^{2n-1}$

and
$$\begin{array}{ccc} \mu^{-1}(0) = S^{2n-1} & \hookrightarrow & \mathbb{C}^* \cdot \mu^{-1}(0) = \mathbb{A}^n - \{0\} = X^{ss} \\ \downarrow & & \downarrow \\ \mu^{-1}(0)/S^1 = S^{2n-1}/S^1 & \xrightarrow[\sim]{\text{homeo.}} & X/\!/\mathbb{C}^* = \mathbb{P}^{n-1} \end{array}$$

§2 Outline of the proof

For (i), (ii) and (iv), the statement for $X \subseteq \mathbb{P}^n$ follows from that for $\mathbb{P}^n$; hence, we assume $X = \mathbb{P}^n$ for now.

Recall: $(\mathbb{P}^n, \omega_{FS})$ arises as a symplectic reduction of $S^1 \curvearrowright (\mathbb{C}^{n+1}, \omega = \mathrm{Im} H)$ where $H : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$, $(z, w) \mapsto z w^*$ is the standard Hermitian I.P. (H is K-invariant as K acts unitarily.)

Let $\|-\|$ be the corresponding norm.

Then $\mu : \mathbb{P}^n \to \mathfrak{k}^*$ is given by $\mu(v) \cdot A = \frac{H(Av, v)}{2i \|v\|^2}$.

Let $v \in \mathbb{C}^{n+1} - \{0\}$; then $p_v : G \to \mathbb{R}$, $g \mapsto \|g \cdot v\|^2$ is constant on K.

<u>Lemma</u> (a) $g \in G$ is a critical point of $p_v \Leftrightarrow \mu(g \cdot [v]) = 0$.

(b) $G \cdot v$ is closed $\Leftrightarrow p_v$ has a critical pt $\Leftrightarrow p_v$ has a minimum.

<u>Pf</u> (a): Let e be the identity of G; then $p_v(g) = p_{g \cdot v}(e)$ and so g is a critical pt of $p_v \Leftrightarrow e$ is a critical pt of $p_{g \cdot v}$.

Take $0 + iA \in \mathfrak{k} \oplus i\mathfrak{k} = \mathfrak{k}_{\mathbb{C}} = \mathfrak{g}$. Then

$$d_e p_v(iA) = \frac{d}{dt} \|\exp(itA) \cdot v\|^2 \Big|_{t=0} = H(iAv, v) + H(v, iAv)$$

$$= 2i\, H(Av, v) \quad \} \text{ as H is K-invariant}$$

$$= -4 \|v\|^2 \mu([v]) \cdot A \quad \text{by defn of } \mu.$$

Hence e is a critical point of $p_v \Leftrightarrow \mu([v]) = 0$.

b) "$\Rightarrow$" If $G \cdot v$ is closed, then $\|G \cdot v\|^2$ is closed, as $\|-\|^2$ is proper.

Hence, $\inf_{g \in G} p_v(g) \in \|G \cdot v\|^2 = \text{Image } p_v$ ie p_v attains its minimum.

Any minimum is a critical point of p_v.

"$\Leftarrow$": An easy calculation shows the 2^{nd} order derivatives of p_v are non-negative, i.e. p_v is convex, so any critical point of p_v is a minimum.

For the leftmost "$\Leftarrow$": suppose $G\cdot v$ is not closed.

Let $G\cdot u$ be a closed orbit in $\overline{G\cdot v}$; then by the theorem of Kempf (talk 1), $\exists$ 1-PS $\lambda: \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t)\cdot v \in G\cdot u$.

By conjugating λ, we can assume $\lambda(S^1) \subseteq K$.

The action of $\lambda(\mathbb{G}_m)$ on $V = \mathbb{C}^{n+1}$ is completely reducible

i.e. $V = \bigoplus_{r\in\mathbb{Z}} V_r$ where $V_r = \{w \in V : \lambda(t)\cdot v = t^r v\}$

and this decomposition is orthogonal w.r.t. H.

Let $v = \sum_r v_r$; then $\underset{\substack{\cap\!\shortmid \\ \overline{G\cdot v} - G\cdot v}}{\lim_{t\to 0}\lambda(t)\cdot v \in G\cdot u} \Rightarrow \begin{cases} v_r = 0 \ \forall r<0 \\ \exists r>0 \text{ s.t. } v_r \neq 0. \end{cases}$ $(*)$

For $A = \frac{d}{dt}\lambda(\exp 2it)\Big|_{t=0} \in \mathfrak{k}$, we have $Av_r = 2irv_r$.

infinitesimal action

Then $H(Av, v) = \sum_{r,s} H(\underbrace{Av_r}_{= 2irv_r}, v_s) = \sum_r 2ir\, H(v_r, v_r)$

decomposition is H-orthogonal

and $\mu([v])\cdot A = \frac{H(Av,v)}{2i\|v\|^2} = \frac{1}{\|v\|^2}\sum_{r\geq 0} r\,\underbrace{H(v_r,v_r)}_{\geq 0} > 0$ by $(*)$.

$\overset{(a)}{\Rightarrow}$ e is not a critical point of p_v

Similarly, e is not a critical point of $p_{g\cdot v}$ $\forall g\in G$ $\boxtimes$

Consequently, this proves the first statement in (ii):

$g\cdot[v] \in \mu^{-1}(0) \overset{(a)}{\Longleftrightarrow} p_v$ has a critical point at g $\overset{(b)}{\Longleftrightarrow} G\cdot v \subseteq \mathbb{A}^{n+1}_{\mathbb{C}}$ is closed

$\Updownarrow$ extension of topological criterion (talk 1).

$[v] \in (\mathbb{P}^n)^{ps}$.

We then deduce (i):

$x \in (\mathbb{P}^n)^{ss} \Longleftrightarrow \exists$ polystable orbit in $\overline{G\cdot x}$ $\overset{(ii)}{\Longleftrightarrow}$ $\overline{G\cdot x} \cap \mu^{-1}(0) \neq \emptyset$.

For the second statement in (ii), we note if $x \in \mu^{-1}(0)$, then $K \cdot x \in \mu^{-1}(0)$ by K-equivariance of μ.

To show $G \cdot x \cap \mu^{-1}(0)$ is a unique K-orbit, one uses the Cartan decomposition of G and convexity of p_v.

(iii): The inclusion

$$\begin{tikzcd} \mu^{-1}(0) \arrow[r, hook] \arrow[d, "p"] & X^{ss} \arrow[d, "\pi"] \\ \mu^{-1}(0)/K \arrow[r] & X/\!/G \end{tikzcd}$$

induces a continuous map

as $\mu^{-1}(0) \to X/\!/G$ is K-invariant.

As a set, $X/\!/G \cong X^{ps}/G$ where $X^{ps} \subseteq X^{ss}$ is the polystable locus

$\uparrow$

(As the closure of each semistable orbit contains a ! orbit which is closed in X^{ss}.)

By (ii) every polystable orbit meets $\mu^{-1}(0)$ in a ! K-orbit; therefore

$$\mu^{-1}(0)/K \cong X^{ps}/G \cong X/\!/G \text{ as sets.}$$

A continuous bijection from a compact space $\mu^{-1}(0)/K$ to a Hausdorff space $X/\!/G$ is a homeomorphism.

iv) As μ lifts the infinitesimal action, it follows that:

$$0 \text{ is a regular value of } \mu \iff K_x \text{ is finite } \forall\, x \in \mu^{-1}(0)$$

$$\Updownarrow \quad \mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$$

$$X^{ss} = X^{ps} = X^{s} \iff \begin{array}{c}\text{every polystable} \\ \text{orbit has a zero} \\ \text{dimensional} \\ \text{stabiliser}\end{array} \overset{(ii)}{\iff} G_x \text{ is finite } \forall\, x \in \mu^{-1}(0)$$

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§3 A comparison of the stratifications

As above, let $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ be a smooth complex proj. variety and $G = K_{\mathbb{C}}$ be a reductive group acting linearly on X such that the compact group K acts unitarily.

Recall for a K-invariant norm $\|-\|$ on $\mathfrak{k}$, the norm square of the moment map $\|\mu\|^2 : X \to \mathbb{R}$ induces a Morse theoretic stratification $X = \bigsqcup_{K\cdot\beta} S^M_{K\cdot\beta}$ (talk 2).

The norm $\|-\|$ then determines a conjugation invariant norm on 1-PS of G as follows:

for $\lambda : \mathbb{G}_m \to G$ a 1-PS of G, $\exists\ g \in G$ s.t
$\lambda' := g\lambda g^{-1}(S^1) \subseteq K$

Consider $d\lambda' : \underset{\substack{|||\\ 2\pi i\mathbb{R}}}{\mathrm{Lie}\, S^1} \to \mathfrak{k}$ and let $\|\lambda\| := \|d\lambda'(2\pi i)\|$.

Recall that the action of G on $X \subseteq \mathbb{P}^n$ and the norm $\|-\|$ on 1-PSs of G determine the Hesselink stratification

$X = \bigsqcup_{([\lambda],d)} S^H_{[\lambda],d}$ into G-invariant strata.

conj class of 1-PS; $d \in \mathbb{R}_{\leq 0}$

with lowest stratum $S_0 = X^{ss}$ (talk 1).

Theorem (Kirwan; Ness)

Let $G = K_{\mathbb{C}}$ be a reductive group acting on a smooth projective variety $X \subseteq \mathbb{P}^n$. For a K-invariant norm $\|-\|$ on $\mathfrak{k}$, the following stratifications coincide:

(1) The Morse theoretic stratification $X = \bigsqcup_{K\cdot\beta} S^M_{K\cdot\beta}$ obtained from $\|\mu\|^2 : X \to \mathbb{R}$

(2) The GIT stratification of Hesselink $X = \bigsqcup_{([\lambda],d)} S^H_{[\lambda],d}$ by adapted 1-PS in the sense of Kempf.

Overview of the proof:

(a) The correspondence between the indices:

$$([\lambda], d) \longleftrightarrow \beta$$

The indices in (1) correspond to <u>rational</u> elements in a Weyl chamber $\mathfrak{t}_+$ for a maximal torus $T \subseteq K$ (as they are closest points to zero of a convex hull of T-weights).

Hence, $\exists!$ minimal $n \in \mathbb{N}$ such that $n\beta$ is <u>integral</u>

<u>ie.</u> $n\beta$ determines a group homomorphism $\begin{array}{l} S^1 \longrightarrow K \\ e^{2\pi i \theta} \mapsto \exp(n\beta\theta) \end{array}$.

By complexifying this we obtain a 1-PS

$\lambda_\beta : \mathbb{G}_m \longrightarrow G$ & let $d_\beta = -\|\beta\|$.

Can also define $([\lambda], d) \to \beta$.

(b) Kirwan's alternative description of $S^M_{K\cdot\beta}$:

$x \in S^M_{K\cdot\beta} \iff \beta \in \mathfrak{t}_+$ is the unique closest point to 0 in $\mu(\overline{G\cdot x}) \cap \mathfrak{t}_+$

In particular, the Morse strata are G-invariant

(The finite time negative gradient flow under $\|\mu\|^2$ is contained in the G-orbit of the point.)

(c) The Kempf-Ness Theorem $\Rightarrow$ the lowest strata agree.

$$S^M_0 \underset{(b)}{=} \{x : 0 \in \mu(\overline{G\cdot x})\} \underset{K\cdot N}{=} X^{ss} = S^H_0.$$

(d) Inductively show higher strata agree using the (c) & the structure of these strata.

For $\beta \neq 0$, we have $S^M_{K\cdot\beta} = G p_\beta^{-1}(Z_\beta^{min})$ [Kirwan]

where Z_β^{min} = minimal Morse stratum for $\|\mu - \beta\|^2$ on Z_β

and $p_\beta : X \to Z_\beta$ is a retraction.

<u>Note</u>: In general, $C_\beta \subsetneq Z_\beta^{min}$.

Similarly $S^H_{[\lambda],d} = G p_\lambda^{-1}(Z_{\lambda,d})$ where $Z_{\lambda,d}$ is a GIT semistable set.

The proof follows by showing $Z_\beta^{min} \overset{(c)}{=} Z_{\lambda_\beta, d_\beta}$. ∎