Stratifications for group actions and moduli problems

Talk 2: Symplectic reduction & moment map stratifications

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§1 Symplectic Actions

Let (M, ω) be a symplectic manifold

i.e. M is a smooth manifold & ω is a closed non-deg. 2-form.
$\Downarrow$ $\dim_{\mathbb{R}} M = \text{even}.$

Eg • $(M, \omega) = (\mathbb{C}^n, \omega = \operatorname{Im} H)$
$\uparrow$ Hermitian inner product $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$.

• $(M, \omega) = (\mathbb{P}^n_{\mathbb{C}}, \omega_{FS})$ Fubini-Study form.

Both examples are Kähler mflds ie $\exists$ cx structure & Riemannian metric compatible with ω.

Let K be a compact Lie group.

Def^n: A (smooth) action of $K \curvearrowright (M, \omega)$ is symplectic if it preserves ω.

The infinitesimal action of K on M is the Lie algebra homomorphism

$$\mathfrak{k} = \operatorname{Lie} K \to \operatorname{Vect}(M) = \Gamma(TM)$$
$$A \mapsto M_A \text{ where } M_{A,m} = \frac{d}{dt} \exp(tA)\cdot m \Big|_{t=0} \in T_m M$$

Contraction with ω gives an iso:

$$\Gamma(TM) \xrightarrow{\sim} \Gamma(T^*M) = \Omega^1(M)$$
$$X \longmapsto \omega(X, -).$$

Idea of symplectic reduction: For $K \overset{\text{sympc}}{\curvearrowright} (M, \omega)$, we want a symplectic quotient.

Note: For dimension reasons alone, M/K may not be symplectic.

Instead we construct a quotient of $\dim = \dim M - 2\dim K$ using a 'moment map' for the action.

Def^n: A lift of the inf. action to a Lie algebra homomorphism is a comoment map.

$$\begin{array}{ccc} \mathfrak{k} & \xrightarrow{\text{inf.}} & \Gamma(TM) \overset{\omega}{\cong} \Omega^1(M) \\ & \searrow^{\Phi} \;\circlearrowleft & \uparrow d \\ & & \Omega^0(M) \end{array}$$

de Rham complex

Dually we have the notion of a moment map.

Def^n: A <u>moment map</u> is a smooth K-equivariant map $\mu: M \to \mathfrak{k}^*$ satisfying $d\mu_A = \omega(M_A, -)$ $\forall A \in \mathfrak{k}$, where $\mu_A : M \to \mathbb{R}$, $m \mapsto \mu(m)\cdot A$

<u>Rmk</u> A moment map may not always exist and is not always unique. (see Ex 1) below)

<u>Ex</u> 1) Let $K \curvearrowright (\mathbb{C}^n, \operatorname{Im} H)$ via a unitary representation $\rho: K \to U(n)$

Then there is a moment map $\mu: \mathbb{C}^n \to \mathfrak{k}^*$ given by

$$\mu(z)\cdot A = \frac{1}{2i} H(\rho_*(A)z, z) = \frac{1}{2}\omega(Az, z)$$

$$\begin{aligned} d_z\mu_A(v) &= \frac{d}{dt}\frac{1}{2i} H(\rho_*(A)(z+tv), z+tv)\Big|_{t=0} \\ &= \frac{1}{2i}\left[H(\rho_*(A)z, v) + H(\rho_*(A)v, z)\right] \\ &= \frac{1}{2i}\left[H(\rho_*(A)z, v) - H(v, \rho_*(A)z)\right] \\ &= \omega(Az, v). \end{aligned}$$

since H is $U(n)$-inv.

For a central element $\chi \in \mathfrak{k}^*$, we also have a moment map $m \mapsto \mu(m) + \chi$. ⇝ moment map is not unique.

2) Let $K \curvearrowright (\mathbb{P}^n_{\mathbb{C}}, \omega_{FS})$ via $\rho: K \to U(n+1)$.

Then there is a moment map $\mu: \mathbb{P}^n_{\mathbb{C}} \to \mathfrak{k}^*$ given by

$\mu(z)\cdot A = \dfrac{\operatorname{Tr}(\tilde{z}^* \rho_*(A)\tilde{z})}{2i\|\tilde{z}\|^2}$ where $\tilde{z} \in \mathbb{C}^{n+1} - \{0\}$ lies over $z \in \mathbb{P}^n_{\mathbb{C}}$.

§2 <u>Symplectic Reduction</u>

Def^n: For $\chi \in \mathfrak{k}^*$, we let $K_\chi \subseteq K$ be the stabiliser of χ for the coadjoint action $K \curvearrowright \mathfrak{k}^*$.

By equivariance of μ, $K_\chi \curvearrowright \mu^{-1}(\chi)$ and we call the topological quotient $\mu^{-1}(\chi)/K_\chi$ the <u>symplectic reduction at χ</u>.

<u>Theorem</u> (Marsden-Weinstein-Meyer)

If $K_\chi \curvearrowright \mu^{-1}(\chi)$ freely, then $\mu^{-1}(\chi)/K_\chi$ has a unique structure of a symplectic manifold such that $\pi: \mu^{-1}(\chi) \to \mu^{-1}(\chi)/K_\chi$ is smooth & the form ω' satisfies $\pi^*\omega' = \iota^*\omega$ for $\iota: \mu^{-1}(\chi) \hookrightarrow M$.

<u>Sketch of proof</u>: $K_x \curvearrowright \mu^{-1}(x)$ freely $\Rightarrow$

- x is a regular value of μ (by inf. lifting property) and so $\mu^{-1}(x) \subseteq M$ is a closed submanifold.
- By the slice theorem: $K_x \curvearrowright \mu^{-1}(x)$ freely & properly (K is compact)
  $\Rightarrow \mu^{-1}(x)/K_x$ has a unique smooth structure such that
  $\pi : \mu^{-1}(x) \to \mu^{-1}(x)/K_x$ is a principal K-bundle.
- We have a short exact sequence

$$0 \to T_m(K_x \cdot m) \longrightarrow T_m \mu^{-1}(x) \longrightarrow T_{\pi(m)}(\mu^{-1}(x)/K_x) \to 0$$

$T_m(K_x \cdot m)$: isotropic subspace $\Downarrow$

$T_m \mu^{-1}(x) = T_m(K_x \cdot m)^{\omega_m}$ symplectic complement, and $T_{\pi(m)}(\mu^{-1}(x)/K_x) \cong \dfrac{T_m(K_x \cdot m)^{\omega_m}}{T_m(K_x \cdot m)}$

$\exists$ induced symplectic form on quotient $\dfrac{T_m(K_x \cdot m)^{\omega_m}}{T_m(K_x \cdot m)}$. ☒

<u>Exercise</u>: x is a regular value of $\mu : M \to \mathfrak{k}^*$ $\iff$ $K_x \curvearrowright \mu^{-1}(x)$ with finite stabilisers.

If x is a regular value, $\mu^{-1}(x)/K_x$ has the structure of a symplectic orbifold.
More generally, $\mu^{-1}(x)/K_x$ has the structure of a stratified symplectic manifold by work of Sjamaar & Lerman.

<u>Ex</u> Let $S^1 \cong U(1) \curvearrowright \mathbb{C}^n$ by scalar multiplication.
The action is symplectic for $\omega = \operatorname{Im} H$ where $H(z,w) = zw^*$ and has moment map $\mu : \mathbb{C}^n \to \mathbb{R} \cong \operatorname{Lie}(S^1)^*$ given by

$$\mu(z_1, \dots, z_n) = \frac{1}{2} \sum_{k=1}^{n} |z_k|^2.$$

For $x = \frac{1}{2}$, we have $K_x = S^1$ and $\mu^{-1}(x) = S^{2n-1} \subseteq \mathbb{C}^n$ and $\mu^{-1}(x)/S^1 = S^{2n-1}/S^1 \cong \mathbb{P}^{n-1}$.
The induced symplectic form is the Fubini-Study form ω_{FS}.
For $x = 0$, we have $\mu^{-1}(x) = \{0\}$ and $\mu^{-1}(x)/S^1 = *$.

§3 The norm square of the moment map

Let $K \curvearrowright (M,\omega)$ symplectically with moment map $\mu : M \to \mathfrak{k}^*$.

Choose an inner product on $\mathfrak{k}$ which is invariant under the adjoint action eg. the Killing form.

Let $\|-\|$ denote the associated norm on $\mathfrak{k}$ and $\mathfrak{k}^*$.

Defⁿ: The norm square of the moment map is the smooth function $\|\mu\|^2 : M \to \mathbb{R}$, $m \mapsto \|\mu(m)\|^2$.

Idea: Use the negative gradient flow of $\|\mu\|^2$, to obtain a Morse stratification of M.

Defⁿ: A Morse-Bott function on a smooth Riemannian mfld (M,g) is a smooth function $f: M \to \mathbb{R}$ such that the set of critical points $\mathrm{Crit}(f)$ is a union of connected submanifolds C_β and for each critical submanifold $C_\beta \subseteq \mathrm{Crit}(f)$, the Hessian $\mathrm{Hess}(f) = \nabla df \in \Gamma(T^*M \otimes T^*M)$ is non-degenerate in the normal directions to C_β.

(∇: Levi-Civita connection)

matrix of 2nd order derivatives of f. ↗

More precisely, we use the Riem. metric to obtain a splitting

$$TM|_{C_\beta} = TC_\beta \oplus \mathcal{N}_\beta$$

↖ normal bundle for $C_\beta \subseteq M$

For $m \in C_\beta$, $\mathrm{Hess}_m(f)$ induces a symmetric bilinear form $H^{\mathcal{N}}_m(f)$ on $\mathcal{N}_{\beta,m}$ and $H^{\mathcal{N}}_m(f)$ should be non-degenerate.

The index λ_β of C_β is the index of $H^{\mathcal{N}}_m(f)$ for any $m \in C_\beta$.

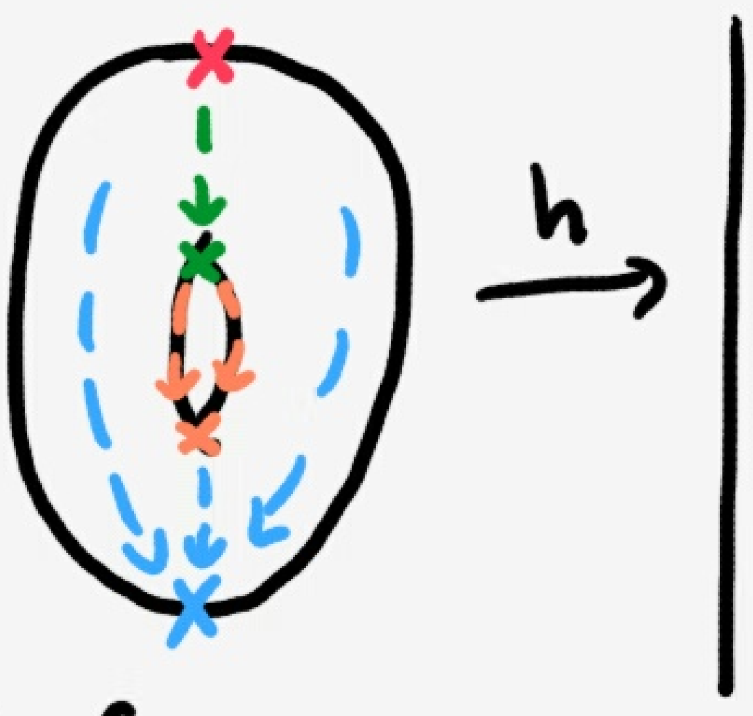
Theorem (Morse-Bott)

The negative gradient vector field $-\nabla f$ of a Morse-Bott funct. f on a compact mfld M induces a Morse stratification $M = \bigsqcup S_\beta$ where $\beta > \beta'$ if $f(C_\beta) > f(C_{\beta'})$ and $\overline{S_\beta} \subseteq \bigcup_{\beta' \geq \beta} S_{\beta'}$.

This stratification is constructed by considering the limit of the negative gradient flow.

More precisely, let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the 1-parameter subgroup of diffeomorphisms of M generated by $-\nabla f$. Then

$$S_\beta := \{ m \in M : \lim_{t\to\infty} \varphi_t(m) \in C_\beta \}.$$



Ex: Height function on a torus $T=(S^1)^2$ is a Morse function with 4 critical points and 4 Morse strata.

Defⁿ : A Morse-Bott function is perfect if

$$P_t(M) := \sum_{k=0}^{\dim M} t^k b_k(M) = MB_t(M) = \sum_{C_\beta \subseteq Crit(f)} P_t(C_\beta)\, t^{\lambda_\beta}$$

$\rightsquigarrow$ Can calculate Betti numbers $b_k(M)$ of M from those of simpler submanifolds $C_\beta \subseteq M$.

Theorem (Atiyah)

For a compact Lie gp $K \curvearrowright (M,\omega)$ symplectically with moment map $\mu: M \to \mathfrak{k}^*$, for each $A \in \mathfrak{k}$, $\mu_A : M \to \mathbb{R}$ is a perfect Morse-Bott function. Moreover, the critical submflds $C_\beta \subseteq M$ are symplectic submflds & the Morse indices are even.

Sketch proof : Let T_A be the closure of the subgroup of K generated by $\exp(\mathbb{R}A)$; then T_A is abelian & connected so $T_A = (S^1)^r$.

As μ lifts the infinitesimal action, $Crit(\mu_A) = M^{\exp \mathbb{R}A} = M^{T_A}$.

One can take a K-inv. Riem. metric g on M (by averaging any metric over K)

$\Rightarrow \exists$ compatible K-inv almost cx structure J defined by $\omega(X,Y) = h(JX,Y)$.

For $m \in M^{T_A}$, one shows $T_m(M^{T_A}) = (T_mM)^{T_A} = \bigcap_{g\in T_A} (T_mM)^g$ is J-inv

$\Rightarrow$ every connected comp. of M^{T_A} is a symplectic submfld.

Moreover the Hessian on the normal directions commutes with $J \Rightarrow$ all e-spaces are J-inv, so the Morse indices are even.

Then μ_A is perfect by the gap criterion (connecting homo is 0 $\Rightarrow$ l.e.s. in coh splits) ☐

Morse type stratification for $\|\mu\|^2 : M \to \mathbb{R}$

Unfortunately $\|\mu\|^2$ is not a Morse-Bott function (for example, $\mathrm{Crit}\|\mu\|^2$ may be singular).

Fortunately Kirwan shows that one can still extend the arguments of Morse theory to $\|\mu\|^2$ to obtain a smooth stratification of M and $\|\mu\|^2$ is "K-equivariantly perfect".

Construction: For $\beta \in \mathfrak{k}$, consider the Morse-Bott function

$$\mu_\beta : M \to \mathbb{R}, \quad \mu_\beta(m) = \mu(m)\cdot\beta.$$

Let $Z_\beta := \mathrm{Crit}(\mu_\beta) \cap \mu_\beta^{-1}(\|\beta\|^2) \leftarrow$ union of connected components of M^{T_β}.

Proposition $\mathrm{Crit}\|\mu\|^2 = \coprod_{K\cdot\beta} C_{K\cdot\beta}$ where $C_{K\cdot\beta} := K\cdot(Z_\beta \cap \mu^{-1}(\beta))$.

Proof: Fix a max^e torus $T \subseteq K$ and +ve Weyl chamber $\mathfrak{t}_+$.
We use $\|-\|$ to identify $\mathfrak{k}^* \cong \mathfrak{k}$ and think of β as an element of either $\mathfrak{k}$ or $\mathfrak{k}^*$. The orbit $K\cdot\beta$ meets $\mathfrak{t}_+$ in a ! pt β.
As $\|\mu\|^2$ is K-invariant, $\mathrm{Crit}\|\mu\|^2$ is K-invariant.
If $m \in \mathrm{Crit}\|\mu\|^2$, $\exists\, k \in K$ s.t $\beta = \mu(k\cdot m) \in \mathfrak{t}_+$.

$k\cdot m$ is critical for $\|\mu\|^2$ $\leftarrow$ (as $M \xrightarrow{\mu} \mathfrak{k}^* \to \mathfrak{t}^*$, $\mu_T : M \to \mathfrak{t}^*$ and $\mu(k\cdot m) = \beta \in \mathfrak{t}$, $\mu(k\cdot m) = \mu_T(k\cdot m)$)

$\Updownarrow$

$k\cdot m$ is critical for $\|\mu_T\|^2 \Leftrightarrow \beta_{k\cdot m} = 0 \Leftrightarrow k\cdot m \in M^{T_\beta} \cap \mu_\beta^{-1}(\|\beta\|^2) = Z_\beta$. ∎

using duality for ω: $d_x\|\mu_T\|^2 = 0 \Leftrightarrow \mu_T(x)_x = 0$ $\mu_\beta(k\cdot m) = \beta\cdot\beta$

Rmk: There are only finitely many critical subsets C_β, as:
$C_\beta \neq \phi \Leftrightarrow \beta$ is the closest pt to 0 in $\mathfrak{t}^+$ of a convex hull of a subset of the T-weights ($= \mu_T(M^T)$ by Atiyah.)

Theorem (Kirwan)

For a compact gp K acting on a compact symplectic manifold M and a K-inv norm $\|-\|$ on $\mathfrak{k}$, the norm square of the moment map $\|\mu\|^2$ induces a finite stratification $M = \coprod S_{K\cdot\beta}$ where $S_{K\cdot\beta} = \{m \in M : \lim_{t\to\infty} \varphi_t(m) \in C_{K\cdot\beta}\}$ are K-inv. & smooth.

(φ_t = neg gradient flow for $\|\mu\|^2$.)

Remarks

- There are only finitely many strata, as there are only finitely many T-weights and, for $\beta \in \mathfrak{t}_+$,

  $$S_{K\cdot\beta} \neq \phi \iff C_{K\cdot\beta} \neq \phi \iff \beta \text{ is the ! closest point in } \mathfrak{t}_+ \text{ to the convex hull of a subset of the T-weights on } M.$$

- We refer to this a Morse stratification for $\|\mu\|^2$; although $\|\mu\|^2$ is not usually a Morse-Bott function, as $C_{K\cdot\beta}$ is often singular.

- The lowest (open) stratum is indexed by $\beta = 0$. We have $S_0 \supseteq C_0 = K\cdot(Z_0 \cap \mu^{-1}(0)) = \mu^{-1}(0)$, as $Z_0 = M$ ($\mu_0 : M \to \mathbb{R}$ is the zero map).

- The stratification is K-equivariantly perfect

  <u>ie</u> $$P_t^K(M) := \sum_{n\geq 0} t^n b_n^K(M) = BM_t^K(M) = \sum_{\substack{\text{critical} \\ \text{loci } C_{K\cdot\beta}}} t^{\lambda_\beta} P_t^K(C_{K\cdot\beta})$$

  where $b_n^K(M) = \dim H_K^n(M;\mathbb{Q})$.

In fact, Kirwan was interested in calculating the cohomology of the symplectic reduction $\mu^{-1}(0)/K$, and, if K acts freely on $\mu^{-1}(0)$, then

$$H^*(\mu^{-1}(0)/K) \cong H_K^*(\mu^{-1}(0)).$$

Hence, the Betti numbers of $\mu^{-1}(0)/K$ can be determined from the K-equivariant Betti numbers of M and the higher strata $S_{K\cdot\beta}$ for $\beta \neq 0$.