Stratifications for group actions & moduli problems

Talk I: Geometric Invariant Theory & instability stratifications

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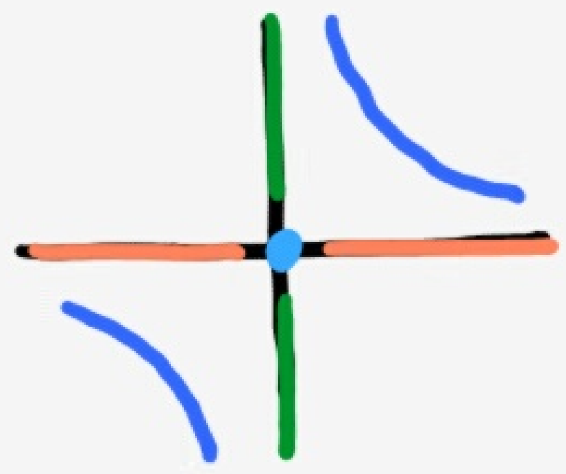
§1 Mumford's Geometric Invariant Theory (GIT)

Consider an action of an affine algebraic group G on a variety X over $k = \mathbb{C}$.

<u>Ex1</u> $G = \mathbb{G}_m \curvearrowright X = \mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$

<u>Orbits</u>:



- conics: for $\alpha \neq 0$ $C_\alpha = \{(x,y) : xy = \alpha\}$
- the origin

} closed

- punctured x-axis
- punctured y-axis

} open

<u>Def</u>

Def^n: A <u>categorical quotient</u> of $G \curvearrowright X$ is a universal G-invariant morphism $\varphi : X \to Y$ (<u>ie</u> $\forall$
$$\begin{tikzcd} X \arrow[r] \arrow[d, "G\text{-inv}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[dl, dashed, "\exists!"] \\ Z & \end{tikzcd}$$
)

<u>Affine GIT quotient</u>

Suppose G is reductive ($\Leftrightarrow G = K_{\mathbb{C}}$ for $K \subseteq G$ max^e compact)
X is affine

$G \curvearrowright \mathcal{O}(X)$ by $(g \cdot f)(x) = f(g^{-1} \cdot x)$.

<u>Hilbert/Nagata</u>: $\mathcal{O}(X)^G$ is a finitely gen. k-algebra.

<u>Mumford</u>: $\mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ induces a morphism of affine varieties $X \xrightarrow{\pi} X/\!/G := \operatorname{Spec} \mathcal{O}(X)^G$.

The affine GIT quotient $X \to X/\!/G$ is a good & categorical quotient.

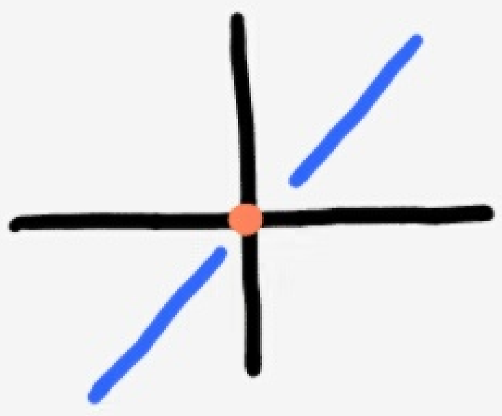
<u>Notation</u>: We write $X/\!/G$, as it is not an orbit space in general.

<u>Ex1</u> $\mathcal{O}(\mathbb{A}^2)^{\mathbb{G}_m} = k[xy]$ and $\mathbb{A}^2 \xrightarrow{\pi} \mathbb{A}^1 = \mathbb{A}^2/\!/\mathbb{G}_m$ is a categorical quotient, but not an orbit space, as $\pi^{-1}(0) =$ union of 3 orbits.

Rmk The preimage of each point under π contains a ! closed orbit.

Ex2 $\mathbb{G}_m \curvearrowright \mathbb{A}^n$ by scalar multiplication.

orbits
- punctured lines through 0 (open)
- the origin (closed)



$\mathcal{O}(\mathbb{A}^n)^{\mathbb{G}_m} = k$ and $\pi: \mathbb{A}^n \longrightarrow *$ contracts all orbits.

$\rightsquigarrow$ We'd like to remove the origin & get $\mathbb{A}^n - \{0\} \longrightarrow \mathbb{P}^{n-1}$. This can be done by introducing a notion of semistability associated to a "linearisation" of the action.

§2 Projective GIT

Now suppose $X \subseteq \mathbb{P}^n$ is a projective variety and G acts linearly on X ie via a linear rep. $G \to GL_{n+1}$.

The homogeneous coord ring replaces the coordinate ring

$$R(X) = k[x_0, \dots, x_n] / \underbrace{I(X)} = \bigoplus_{r \geq 0} R(X)_r$$

graded by homogeneous degree.

homogeneous ideal for $X \subseteq \mathbb{P}^n$

As the action is linear, it respects the grading and

$$R(X)^G = \bigoplus_{r \geq 0} R(X)_r^G \hookrightarrow R(X) = \bigoplus_{r \geq 0} R(X)_r$$

finitely generated

induces a rational morphism of projective varieties

$$\begin{array}{ccc} X & \dashrightarrow & X/\!/G := \operatorname{Proj} R(X)^G \\ \text{open } \cup\!| & \nearrow_{\pi} & \\ X^{ss} & & \end{array}$$

semistable locus

π is the projective GIT quotient; it's a good and categorical quotient of $G \curvearrowright X^{ss}$.

Def^n: $x \in X$ is <u>semistable</u> if $\exists\, f \in R(X)_r^G$ for $r > 0$ such that $f(x) \neq 0$.

Notation: X^{ss} is the set of semistable points.

Rmk $\pi : X^{ss} \longrightarrow Y := X/\!/G$ can be constructed by gluing affine GIT quotients $\pi_f : X_f \to Y_f$ for $f \in R(X)^G_+$. In fact, $X^{ss} = \bigcup_{f \in R(X)^G_+} X_f$.

Ex3 $\mathbb{G}_m \curvearrowright \mathbb{P}^n$ by $\mathbb{G}_m \to GL_{n+1}$, $t \mapsto \mathrm{diag}(t^{-1}, t, \dots, t)$
$R(\mathbb{P}^n) = k[x_0, \dots x_n] \supseteq R(\mathbb{P}^n)^{\mathbb{G}_m} = k[x_0x_1, x_0x_2, \dots x_0x_n]$ &
$(\mathbb{P}^n)^{ss} = \{[x_0 : \ : x_n] : x_0x_i \neq 0 \text{ for some } i\} \cong \mathbb{A}^n - \{0\}$.
The proj. GIT quotient is $(\mathbb{P}^n)^{ss} \xrightarrow{\pi} \mathbb{P}^n/\!/\mathbb{G}_m = \mathrm{Proj}\, k[x_0x_1, \dots, x_0x_n] \cong \mathbb{P}^{n-1}$.

§3 The Hilbert-Mumford criterion for semistability

Problem: To determine X^{ss} we need to know $R(X)^G$
hard to compute.

Topological Criterion for semistability

For $x \in X \subseteq \mathbb{P}^n$ choose a lift $\tilde{x} \in \mathbb{A}^{n+1} - \{0\}$.
Then x is semistable $\iff 0 \notin \overline{G \cdot \tilde{x}}$.

Pf: "$\Rightarrow$" $0 \in \overline{G \cdot \tilde{x}} \Rightarrow$ every $f \in R(X)^G_+$ satisfies $f(x) = f(0) = 0$.
"$\Leftarrow$" $\tilde{X}$ is affine & G is geometrically reductive $\Rightarrow \exists$ G-inv homogeneous $f \in \mathcal{O}(\tilde{X})^G_r = R(X)^G_r$ for $r > 0$, that distinguishes the disjoint closed G-inv sets 0 & $\overline{G \cdot \tilde{x}}$
ie $f(0) = 0 \neq f(\overline{G \cdot \tilde{x}}) = f(x)$. ∎

Q: How do we study an orbit closure?
$\to$ Use limits of 1-parameter subgroups $\lambda : \mathbb{G}_m \to G$.

Valuative Criterion for properness: X is proper.

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\lambda(-)\cdot x"] \arrow[d, hook] & X \arrow[d] \\ \mathbb{A}^1 \arrow[r] \arrow[ur, dashed, "\exists !"] & * \end{tikzcd}$$

Let $\lim_{t \to 0} \lambda(t) \cdot x$ denote the image of 0, and similarly $\lim_{t \to \infty} \lambda(t) \cdot x = \lim_{t \to 0} \lambda^{-1}(t) \cdot x$.

Note: for $\tilde{x} \in \tilde{X}$, the limits $\lim_{\substack{t\to 0 \\ \text{or } \infty}} \lambda(t)\cdot\tilde{x}$ may not exist.
not proper

If $\exists$ 1-PS $\lambda : \mathbb{G}_m \to G$ s.t. $\lim_{t\to 0} \lambda(t)\cdot\tilde{x} = 0 \Rightarrow x$ is not semistable.

Conversely, we can apply:

Theorem (Kempf, Richardson)

Let $\underset{\text{red}}{G} \curvearrowright \underset{\text{closed}}{V \subseteq \mathbb{A}^n}$ linearly. For $v \in V$ and any G-inv closed $Z \subseteq V$ which meets $\overline{G\cdot v}$, $\exists$ 1-PS $\lambda : \mathbb{G}_m \to G$ s.t $\lim_{t\to 0} \lambda(t)\cdot v \in Z$.

Thm (Hilbert-Mumford criterion)

$x \in X$ is semistable $\iff \forall$ 1-PS $\lambda : \mathbb{G}_m \to G$, $\lim_{t\to 0} \lambda(t)\cdot\tilde{x} \neq 0$.

Defn: Let $\mu = \mu(x,\lambda)$ be the unique integer such that $\lim_{t\to 0} t^{\mu}\lambda(t)\cdot\tilde{x}$ exists and is non-zero.

$\mu(x,\lambda)$ is called the Hilbert-Mumford weight.

Rmk • $\mu(x,\lambda)$ = weight of the $\lambda(\mathbb{G}_m)$-action on $\mathcal{O}_X(1)$ over the fixed point $\lim_{t\to 0} \lambda(t)\cdot x \in X$.

- $\mu(x,\lambda) > 0 \Rightarrow \lim_{t\to 0} \lambda(t)\cdot\tilde{x}$ does not exists,

  $\mu(x,\lambda) = 0 \Rightarrow \lim_{t\to 0} \lambda(t)\cdot\tilde{x}$ exists and is non-zero,

  $\mu(x,\lambda) < 0 \Rightarrow \lim_{t\to 0} \lambda(t)\cdot\tilde{x} = 0$.

Thm (Hilbert-Mumford criterion - version 2)

$x \in X$ is semistable $\iff \mu(x,\lambda) \geq 0 \ \forall$ 1-PS $\lambda : \mathbb{G}_m \to G$.

Rmk: Instead of considering an embedding $X \subseteq \mathbb{P}^n$ s.t. the G-action on X is linear, we can use a linearisation i.e. an ample line bundle $\pi : L \to X$ with a G-action by bundle autos

$$\begin{array}{ccc} G\times L & \to & L \\ \downarrow & & \downarrow \\ G\times X & \to & X. \end{array}$$

Then $R(X)$ is replaced by $R(X,L) = \bigoplus_{r\geq 0} H^0(X, L^{\otimes r})$ & we have $X^{ss}(L) \to X/\!/_L G \cong \operatorname{Proj} R(X,L)^G$.

§4 Instability Stratifications

Idea: Stratify the unstable locus $X^{us} = X - X^{ss}$ by associating to each $x \in X^{us}$ a 1-PS that is "most responsible" for x being unstable, using a normalised Hilbert-Mumford weight.

Note: $\inf_\lambda \mu(x,\lambda)$ does not exist as $\mu(x,\lambda^n) = n\mu(x,\lambda)$ for $n \in \mathbb{N}$,

but $\inf_\lambda \frac{\mu(x,\lambda)}{\|\lambda\|}$ does exist for a norm $\|-\|$ on 1-PSs.

Fix $T \subseteq G$ max^{e} torus and $\|-\|$ a Weyl-invariant norm on the space of 1-PS $X_*(T)_{\mathbb{R}}$. For any $\lambda : \mathbb{G}_m \to G$, let $\|\lambda\| := \|g\lambda g^{-1}\|$ for g s.t. $g\lambda g^{-1} \in X_*(T)$.

Eg a) Euclidean norm for $G = GL_n$

T = diagonal matrices, the Euclidean norm on $X_*(T)_{\mathbb{R}} = \mathbb{R}^n$ is S^n-invariant.

b) $G = GL_{n_1} \times \dots \times GL_{n_r} \rightsquigarrow$ weight each Euclidean norm using $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{N}^r$

$$\|(\lambda_1, \dots, \lambda_r)\|_\alpha^2 := \sum_{i=1}^r \alpha_i \|\lambda_i\|^2.$$

Let $M(x) := \inf_\lambda \frac{\mu(x,\lambda)}{\|\lambda\|}$; then $M(x) \geqslant 0 \iff x \in X^{ss}$

Def^n: A primitive 1-PS λ is adapted to x if $M(x) = \frac{\mu(x,\lambda)}{\|\lambda\|}$.

Let $\Lambda(x) := \{$1-PS adapted to $x\}$.

Theorem (Kempf)

Let $x \in X^{us}$. Then

(i) $\Lambda(x) \neq \phi$; it's a full conj. class for $P_x \leq G$ parabolic subgp.

(ii) $\Lambda(x) = \Lambda(x_0)$ where $x_0 = \lim_{t\to 0} \lambda(t)\cdot x$ and $\lambda \in \Lambda(x)$.

(iii) $\Lambda(g \cdot x) = g\,\Lambda(x)\,g^{-1}$ for $g \in G$.

Moreover $P_x = P(\lambda) := \{g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1}$ exists in $G\}$ for any $\lambda \in \Lambda(x)$.

Def^n: For $[\lambda] = \{ g\lambda g^{-1} : g \in G \}$ and $d \in \mathbb{R}_{<0}$, let

$$S_{[\lambda],d} := \{ x : M(x) = d \text{ and } \exists \, \lambda \in [\lambda] \text{ adapted to } x \}.$$

Fix a representative $\lambda \in [\lambda]$ and let

$$S_{\lambda,d} := \{ x : M(x) = d \text{ and } \lambda \text{ is adapted to } x \}$$

$$Z_{\lambda,d} := \{ x \in X^{\lambda} : M(x) = d \text{ and } \lambda \text{ is adapted to } x \}.$$

Theorem (Hesselink; Kirwan) For $G \curvearrowright X$ proj. var. w.r.t. L, there is a finite Hesselink stratification

$$X^{us} = \bigsqcup_{([\lambda],d)} S_{[\lambda],d}$$ into G-invariant locally closed subschemes of X

such that

- $\overline{S_{[\lambda],d}} \subseteq \bigsqcup_{|d'| \geq |d|} S_{[\lambda'],d'}$
- The indices are determined by the weights of $T \subseteq G$ max^e torus.
- $S_{[\lambda],d} = G S_{\lambda,d}$ and $S_{\lambda,d} = p_{\lambda}^{-1}(Z_{\lambda,d})$ where $p_{\lambda} : X \to X^{\lambda}$ is $x \mapsto \lim_{t \to 0} \lambda(t) \cdot x$.
- $Z_{\lambda,d}$ = GIT semistable set for a smaller reductive gp (the Levi of $P(\lambda)$) acting on $X^{\lambda}_d := X^{\lambda} \cap M^{-1}(d)$ with respect to a modified linearisation.

Notation Write $S_0 = X^{ss} \subseteq X$; this is the lowest (open) stratum.

Applications

- For X smooth, can compute cohomology of $X /\!/ G$ in terms of the G-equiv coh. of the strata – see Kirwan's thesis.
- These stratifications can be used to describe birational transformations in variation of GIT [Dolgachev-Hu & Thaddeus].
  $\rightsquigarrow$ semi-orthogonal decompositions in derived categories of stack quotients [Ballard-Favero-Katzarkov & Halpern-Leistner].

Example: Ordered points on $\mathbb{P}^1$ (Kirwan; Newstead)

For $n \geq 1$, let $SL_2 \curvearrowright (\mathbb{P}^1)^n \xhookrightarrow{\text{Segre embedding}} \mathbb{P}^{2^n-1}$

where $SL_2 \curvearrowright \mathbb{P}^1$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x:y] = [ax+by : cx+dy]$.

Fix the diagonal maximal torus $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{G}_m \right\}$

Then any 1-PS of SL_2 is conjugate to a 1-PS in T and T has two primitive 1-PS $\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and $\lambda^{-1}(t) = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$ which are conjugate.

Thus every Hesselink stratum has the form $S_{[\lambda],d}$ for λ as above and $d \in \mathbb{R}$.

We can calculate the strata from the limit sets $Z_{\lambda,d}$ which are contained in the λ-fixed locus of $X = (\mathbb{P}^1)^n$:

$$X^\lambda = \{ x = (p_1, \dots p_n) \in (\mathbb{P}^1)^n : p_i = [1:0] \text{ or } [0:1] \ \forall i \} \leftarrow 2^n \text{ points}$$

We use the following alternative definition of μ:

$$\mu(x, \lambda) = -\min \{ \lambda\text{-weights of } x \}$$

λ acts on $\mathbb{P}^1$ with weights ± 1 and on $(\mathbb{P}^1)^n \hookrightarrow \mathbb{P}^{2^n-1}$ with weights $\{-n, -n+2, \dots, n-2, n\}$.

For $x = (p_1, \dots, p_n) \in X^\lambda$ we have:

$$\mu(x,\lambda) = \begin{cases} -n & \text{if } p_i = [1:0] \ \forall i, \\ -n+2 & \text{if } \exists ! j \text{ s.t } p_j \neq [1:0], \\ \vdots & \vdots \\ n & \text{if } p_i = [0:1] \ \forall i. \end{cases}$$

For $0 \leq j < \frac{n}{2}$, $d_j := -n + 2j < 0$. Then we claim

$$Z_{\lambda, d_j} = X^\lambda_{d_j} = \left\{ x = (p_1, \dots, p_n) \in X^\lambda : \begin{array}{l} \exists I \subseteq \{1, \dots n\} \\ \text{s.t } |I| = n-j \end{array} \ \& \ p_i = \begin{cases} [1:0] & i \in I \\ [0:1] & i \notin I \end{cases} \right\}$$

$$S_{\lambda, d_j} = \left\{ x = (p_1, \dots, p_n) \in X : \begin{array}{l} \exists I \subseteq \{1, \dots, n\} \\ \text{s.t } |I| = n-j \end{array} \ \& \ p_i = [1:0] \text{ iff } i \in I \right\}$$

$$S_{[\lambda], d_j} = \{ x = (p_1, \dots, p_n) \in X : \text{exactly } n-j \text{ of the } p_i \text{ coincide} \}$$

$$\& \ X^{ss} = \{ x = (p_1, \dots, p_n) : \text{no } p_i \text{ appears with multiplicity} > \tfrac{n}{2} \}.$$

To prove this description of the Hesselink strata, we note

$$P(\lambda) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \lim_{t \to 0} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \text{ exists in } SL_2 \right\}$$

$$= \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in SL_2 \right\}$$

and $p_\lambda : X \to X^\lambda$

$$p_\lambda(p_1, \dots, p_n) = (p_\lambda(p_1), \dots\dots p_\lambda(p_n))$$

where $p_\lambda : \mathbb{P}^1 \longrightarrow (\mathbb{P}^1)^\lambda = \{[1:0]\} \cup \{[0:1]\}$

$$[x:y] \longmapsto \begin{cases} [0:1] & \text{if } y \neq 0 \\ [1:0] & \text{if } y = 0. \end{cases}$$

In particular, $\lambda(\mathbb{G}_m) \cong$ Levi subgroup of $P(\lambda)$ and for the modified linearisation on $X^\lambda_{d_j}$, this 1-parameter subgroup is no longer destabilising.

Hence, $Z_{\lambda, d_j} := (X^\lambda_{d_j})^{ss} = X^\lambda_{d_j}$.

Since $S_{\lambda, d_j} := p_\lambda(Z_{\lambda, d_j})$, we get the above description of S_{λ, d_j}.

Finally as $G = SL_2$ acts transitively on $\mathbb{P}^1$ and

$S_{[\lambda], d_j} = G S_{\lambda, d_j}$, we obtain the above description of the unstable strata and the semistable set is the complement to the union of the unstable strata.