# SEMINAR ON THE REPRESENTATION THEORY OF THE SYMMETRIC GROUP

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A representation of a group is an action of the group on a vector space; that is, for each element in the group, we have an automorphism of the vector space, and their compositions are compatible with the group multiplication. In this seminar we will focus on the representation theory of finite groups, and in particular the symmetric group. Many prominent mathematicians have studied the representation theory of the symmetric group, such as Frobenius, Schur and Young. The representation theory of the symmetric group also has strong connections to combinatorics, geometry and topology, as well as applications to other branches of mathematics, such as mathematical physics. Every representation is built out of irreducible representations and the main aim will be to describe these irreducible representations for the symmetric group combinatorially using partitions and certain diagrams, called Young diagrams. This forms the foundations for studying the representation theory of matrix groups like the general linear group (and, more generally, Lie groups).

**Prerequisites:** Students attending this seminar should have completed LAI and LAII.

**Literature:** The main reference is the book of Sagan [3]. Other good sources are [2, 4, 5].

**Guidlines for the talks:**

- The talks are given either in English or German and should last approximately 80 minutes to allow for 10 minutes of questions.
- Participants are expected to discuss their talk with the lecturer the week before they are scheduled to speak (and bring with them a draft of their talk notes).
- The focus of the talk should be on the mathematical arguments, rather than on historical remarks.
- All required definitions and mathematical claims should be clearly stated; in particular, the definitions of all terms in italics in the descriptions below should be given.
- The speaker should make sure that the assumptions and the claim are clear, in order for the other participants to be able to follow proofs and explanations.
- All talks should take place at the blackboard.

## Description of the talks

**Talk 1: Algebraic foundations.** We recall some basic definitions and properties of groups, rings, fields, modules and algebras; this talk is given by the lecturer.

**Talk 2: The symmetric group and its conjugacy classes.** Give the definition of the *symmetric group* $S_n$ and describe the two-line notation and cycle notation for *permutations* $\pi \in S_n$ (provide examples of both notations). Define the *cycle type* of a permutation and explain how this can be encoded using *partitions* (the definition of a partition should also be given). Define the *conjugacy relation* on a group $G$ and show that this is an equivalence relation; then define the *conjugacy classes* to be the equivalence classes. Prove that the conjugacy classes in $S_n$ are in bijection with the set of partitions of $n$ (for more details, see [2, 1.2.1 and 1.2.6]). Prove that the order of the *centralizer group* of a permutation with cycle type given by a partition $\lambda$ can be expressed in terms of $\lambda$ (that is, prove [3, Proposition 1.1.1]). Finally prove that $S_n$ is generated by $n-1$ *transpositions* (see [2, 1.1.16-17]) and, using transpositions, define the *sign* of a permutation, which gives a group homomorphism $\mathrm{sgn} : S_n \to \{\pm 1\}$.
**References.** The primary reference is [3, Section 1.1]; for more details, see also [2, Section 1.1-1.2].

**Talk 3: Group representations and the group algebra.** For a group $G$, give the definition of a *(complex) matrix representation* $\rho : G \to \mathrm{GL}_d$ of $G$ of *degree*[1] $d$; then explain how this defines an *action* of $G$ on $\mathbb{C}^d$. Explain the examples of the *sign representation* and the *defining representation* of $S_n$. Define a *(complex) G-module*, and explain how a (complex) matrix representation of $G$ of degree $d$ determines a

[1]Usually $d$ is referred to as the *dimension* of $\rho$; however, in [3], this is called the degree and one writes $\deg(\rho) = d$

$G$-module structure on $V = \mathbb{C}^d$. Conversely, describe how for a $G$-module $V$, one can define an associated matrix representation $\rho_{\mathcal{B}} : G \to \mathrm{GL}_d$ of $G$ of degree $d := \dim(V)$ by choosing a basis $\mathcal{B}$ of $V$. Explain how the representations $\rho_{\mathcal{B}}$ and $\rho_{\mathcal{C}}$ are related for different choice of bases $\mathcal{B}$ and $\mathcal{C}$ of a $G$-module $V$.

For a group $G$ acting on a finite set $S$, define the associated *permutation representation* as a $G$-module (for $S_n$ acting on the set $\{1, \dots, n\}$, this is the defining representation); for the action of a finite group $G$ on itself by left multiplication, this gives the *(left) regular representation* whose associated $G$-module has the structure of a $\mathbb{C}$-algebra and is called the *group algebra* and denoted $\mathbb{C}[G]$. Another example is the *(left) coset representation* associated to a subgroup $H < G$; explain how the regular representation is a special case of this. Examples of these representations should also be presented. If there is time, explain that a representation of $G$ is the same as a (left) $\mathbb{C}[G]$-module (see [5, Section 1.1])

**References.** The primary reference is [3, Sections 1.2-1.3]; see also [5, Section 1.1].

**Talk 4: Reducibility and Maschke's Theorem.** Define a *$G$-submodule* of a $G$-module and give the definition of a *reducible* (resp. *irreducible*) $G$-module $V$; then explain how the reducibility of $V$ can be equivalently formulated in terms of the matrix representation $\rho_{\mathcal{B}}$ having a block upper triangular form with respect to some basis $\mathcal{B}$ of $V$. Explain when the defining representation of $S_n$ is reducible and when the group algebra $\mathbb{C}[G]$ is reducible.

Recall the notion of *(internal) direct sums* and *linear complements* of vector subspaces; then define the analogous notion of ($G$-)*complements* for $G$-submodules and rephrase this in terms of matrix representations. Explain how one can define a $G$-complement of a $G$-submodule $W \subset V$ using an *inner product* on $V$ that is *invariant under the $G$-action* on $V$ (see [3, 1.5.2]). Then, for a finite group, describe how to produce such an invariant innner product from any inner product on $V$ by averaging over the group and state and prove Maschke's Theorem, which asserts that for a finite group $G$,any non-zero $G$-module decomposes into a direct sum of irreducible representations. Finally introduce the notion of a *completely reducible* $G$-module and rephrase Maschke's Theorem in this language. Examples should be given throughout and, if there is time, an example should be given to show that the finiteness condition on $G$ in Maschke's theorem cannot be dropped.

**References.** The primary reference is [3, Sections 1.4-1.5]; see also [5, Section 1.2].

**Talk 5: Schur's Lemma and the commutant algebra.** Give the definition of a *$G$-homomorphism* and *$G$-isomorphism* between $G$-modules $V$ and $W$; then explain how to rephrase this in terms of the matrix representations associated to bases of $V$ and $W$. Provide some examples (with focus on the symmetric group; however, [3, Example 1.6.3] can be skipped, as it requires some terminology from later on). Prove that the kernel and image of a $G$-homomorphism are $G$-submodules [3, Proposition 1.6.4] and then state and prove Schur's Lemma, which says that any non-zero $G$-homomorphism between irreducibles $G$-modules is an $G$-isomorphism, and provide the corresponding statement for matrix representations [3, Corollary 1.6.7]. State and prove [3, Corollary 1.6.8] for irreducible complex matrix representations.

Define the *commutant algebra* $\mathrm{Com}(\rho)$ of a matrix representation $\rho : G \to \mathrm{GL}_d$ and the *endomorphism algebra* $\mathrm{End}(V)$ of a $G$-module $V$ ( [3, Definition 1.7.1]); then explain why these are $\mathbb{C}$-algebras and show that the commutator algebra of a matrix representation $\rho_{\mathcal{B}}$ associated to a basis $\mathcal{B}$ of a $G$-modules $V$ is isomorphic (as an algebra) to the endomorphism algebra of $V$; that is $\mathrm{Com}(\rho_{\mathcal{B}}) \cong \mathrm{End}(V)$. Finally, describe the commutant algebra of a representation $\rho$ which is a direct sum of two non-isomorphic (resp. isomorphic) representations [3, Examples 1.7.2-3].

**References.** The primary reference is [3, Sections 1.6 & 1.7.1-1.7.3]; see also [5, Section 2.1].

**Talk 6: Tensor products of matrix representations and the commutant algebra.** Define the *tensor product* $X \otimes Y$ of square complex matrices $X \in \mathrm{Mat}_d$ and $Y \in \mathrm{Mat}_e$ as a $de \times de$-matrix [3, Definition 1.7.4]. Recall the definition of the *commutant algebra* $\mathrm{Com}(\rho)$ of a matrix representation $\rho : G \to \mathrm{GL}_d$ from the previous week (see [3, Definition 1.7.1]). For a representation $\rho$, which has a decomposition $\rho = \bigoplus_{i=1}^{k} (\rho_i)^{\oplus m_i}$ where $\rho_i$ are pairwise non-isomorphic irreducible matrix representations as given by Maschke's Theorem, describe $\mathrm{Com}(\rho)$ using tensor products of matrices and compute $\deg(\rho)$ and $\dim \mathrm{Com}(\rho)$ in terms of $m_i$ and $d_i := \deg(\rho_i)$'s (see [3, page 25-26]). Give the definition of an *abstract tensor product* $V \otimes W$ of two (finite dimensional complex) vector spaces $V$ and $W$ as in [1, Definition 3.1] and outline the construction of the tensor product [1, Theorem 3.2]; moreover, explain that if $\mathcal{A} = \{v_1, \dots, v_n\}$ and $\mathcal{B} = \{w_1, \cdots, w_m\}$ are bases of $V$ and $W$, then $\{v_i \otimes w_j : 1 \le i \le n, 1 \le j \le m\}$ is a basis of $V \otimes W$ (thus $\dim(V \otimes W) = \dim(V)\dim(W)$) [1, Theorem 3.3]. If $V = \mathrm{Mat}_d$ and $W = \mathrm{Mat}_e$, explain that there is an isomorphism $\mathrm{Mat}_d \otimes \mathrm{Mat}_e \cong \mathrm{Mat}_{de}$ and that the tensor product of matrices defined above uses this isomorphism.

Define the *center* $Z_A$ of a $\mathbb{C}$-algebra $A$ and prove that $Z_{\mathrm{Mat}_e} \cong \mathbb{C}$ [3, Proposition 1.7.6]. State and prove [3, Theorem 1.7.8] which describes $\mathrm{Com}(\rho)$ and its center; then state the alternative formulation of this result in terms of $G$-modules and endomorphism algebras [3, Theorem 1.7.9] and also state [3, Proposition 1.7.10].
**References.** The primary reference is [3, Section 1.7].

**Talk 7: Group characters and the character table.** Define the *character* $\mathrm{tr}(\rho) : G \to \mathbb{C}$ of a matrix representation $\rho : G \to \mathrm{GL}_d$. For a $G$-module $V$, define the character of $V$ by using a basis of $V$ (and check this is independent of the choice of basis). Give the notions of *irreducible characters* and *linear characters*. Compute the character of the defining representation of $S_n$ and the regular representation $\mathbb{C}[G]$ of a finite group $G$. State and prove [3, Proposition 1.8.5] on the properties of characters. Define the set $R(G)$ of all *class functions* on $G$ and prove that $R(G)$ is a $\mathbb{C}$-vector space and compute $\dim R(G)$ by finding a basis indexed by the conjugacy classes in $G$.

Define the *character table* of a finite group $G$; make the remark that we do not yet know that there are only finitely many irreducible representations of $G$, but we will eventually prove that the number of irreducible representations of $G$ is equal to the number of conjugacy classes of $G$ and so the character table is always finite and square. Compute a few examples of character tables. As a step towards proving that the character table is square, introduce the inner product of two functions $G \to \mathbb{C}$ and prove the inner product of two group characters is given by the formula in [3, Proposition 1.9.2]; then explain the special version of this formula for class functions (see [3, p. 35]).
**References.** The primary reference is [3, Sections 1.8 & 1.9.1-1.9.2]; see also [5, Section 3.1].

**Talk 8: Orthogonality relations for characters.** Start by recalling the definition of the inner product of two group characters from the previous talk. Prove that irreducible characters are orthonormal with respect to this inner product; that is, state and prove [3, Theorem 1.9.3]. Then state and prove [3, Corollary 1.9.4], which says that for a matrix representation $\rho$ with a decomposition $\rho = \bigoplus_{i=1}^{k} (\rho_i)^{\oplus m_i}$ where $\rho_i$ are pairwise non-isomorphic irreducible representations as in Maschke's Theorem, the associated character $\chi = \mathrm{tr}(\rho)$ can be written in terms of the characters $\chi_i = \mathrm{tr}(\rho_i)$, and moreover the inner products of all these characters can be computed, which gives an equivalent formulation of irreducibility of $\rho$. Using these results give a full description of the character table of $S_3$. Compute the character of the complement of the trivial representation $\mathbb{C}\{1 + \cdots + n\}$ in the defining representation of $S_n$. Finally, apply Maschke's Theorem to give a decomposition of the group algebra $\mathbb{C}[G]$ into a direct sum of non-isomorphic irreducible representations with multiplicities and state and prove [3, Proposition 1.10.1] about the properties of this decomposition.
**References.** The primary reference is [3, Sections 1.9 & 1.10.1]; see also [5, Section 3.2].

**Talk 9: Consequences of the orthogonality relations.** State and prove [3, Proposition 1.10.2], which says that the set of irreducible characters of $G$ form a basis for the space $R(G)$ of class functions on $G$; here all the necessary ingredients should be recalled. As a corallary, deduce that the number of irreducible characters of $G$ equals the number of conjugacy classes in $G$, which proves that the character table is finite and square. Then state and prove [3, Theorem 1.10.3], and explain how this gives an alternative method to compute character tables.

Define the tensor product of two matrix representations (for two groups $G$ and $H$), prove that this is a matrix representation of $G \times H$ and compute its character [3, 1.11.1-2]. Explain that when $V$ and $W$ are two modules for groups $G$ and $H$, then the tensor product $V \otimes W$ (as vector spaces) has a natural $G \times H$-module structure and moreover $\rho_{\mathcal{A}\otimes\mathcal{B}} = \rho_{\mathcal{A}} \otimes \rho_{\mathcal{B}}$ for bases $\mathcal{A}$ and $\mathcal{B}$ of $V$ and $W$, where $\mathcal{A}\otimes\mathcal{B}$ is the associated basis of $V \otimes W$. Prove that the irreducible representations of $G$ and $H$ completely determine the irreducible representations of $G \times H$; that is, state and prove [3, Theorem 1.11.3].
**References.** The primary reference is [3, Sections 1.10 & 1.11]; see also [5, Section 3.3].

**Talk 10: Restricted and induced representations.** For a subgroup $H < G$, define the *restriction* to $H$ of a representation of $G$ and prove that this is a representation of $H$. Conversely given a representation of $H$, explain how to define the *induced representation* of $G$ using representations of the set of left $H$-cosets; then prove that this is a representation of $G$ (that is, state and prove [3, 1.12.3-4]). Furthermore, show that this definition is (up to isomorphism) independent of the choice of representatives of these cosets [3, Proposition 1.12.5]. Provide some examples of restriction and induction. Then prove a formula for the character of an induced representation and state and prove Frobeinus reciprocity [3, Theorem 1.12.6], which relates the inner products of restricted and induced characters.
**References.** The primary reference is [3, Section 1.12]; see also [5, Sections 4.2 & 4.3].

**Talk 11: Young tabloids and dominance.** We now focus our attention on the symmetric group $S_n$. Recall that the conjugacy classes in $S_n$ are in bijection with the set of partitions of $n$ and also in bijection with the set of irreducible representations of $S_n$. For a partition $\lambda$ of $n$, define: the *Ferrer's diagram* of shape $\lambda$, the *Young subgroup* $S_\lambda < S_n$ and the induced module $V^\lambda$ associated to this subgroup, a *Young tableau* of shape $\lambda$ and the equivalence relation on Young tableaux of shape $\lambda$ whose equivalence classes are called *Young tabloids* of shape $\lambda$, and then finally the *permutation module* $M^\lambda$ (see [3, 2.1.1-2.1.5] and provide examples). Prove that $M^\lambda$ is a *cyclic* $G$-module and compute its dimension, and then show that $M^\lambda \cong V^\lambda$ as $S_n$-modules [3, Theorem 2.1.12].

Give an overview of our strategy to describe all irreducible representations of $S_n$: since $M^\lambda$ will not be irreducible in general, we plan to find an ordering of partitions of $n$ such that the associated permutation module for the first partition is irreducible and each $M^\lambda$ is a sum of irreducibles appearing in lower permutation modules plus a unique new irreducible representation $S^\lambda$ called the associated Specht module (we will prove that the Specht modules give a full set of irreducible representations of $S_n$). The required ordering on partitions is called *dominance*. First recall the notion of a *partial* (resp. *total*) *order* on a set as a relation. Then define the *dominance relation* on the set of partitions of $n$ and describe how one can construct the associated *Hasse diagram* for this partial order (draw this diagram for some small values of $n$). Prove the dominance lemma for partitions and prove that the *lexicographic order* refines the dominance order [3, 2.2.2-6].
**References.** The primary reference is [3, Sections 2.1 & 2.2].

**Talk 12: Specht modules and the main result.** Recall our strategy to describe all irreducible representations of $S_n$: we plan to prove that the dominance ordering of partitions of $n$ has the property that the associated permutation module for the lowest partition $1^n$ is irreducible and each $M^\lambda$ is a sum of irreducibles appearing in lower permutation modules plus a unique new irreducible representation $S^\lambda$ which is called a Specht module. In this talk we will prove this statement and show that the Specht modules give a full set of irreducible representations of $S_n$. First to define the Specht modules, we define the *row and column stabilizers* of a Young tableau and associated *polytabloid*, prove some propeties of these definitions [3, Lemma 2.3.3], and define the *Specht module* $S^\lambda$ to be the submodule of $M^\lambda$ equal to the span of all polytabloids of shape $\lambda$. Show that $S^\lambda$ is cyclic and the lowest Specht module $S^{(1^n)}$ is the sign representation. (Optional): Calculate the Specht modules for $n = 3$.

To prove our main result (that the Specht modules give a full set of irreducible representations of $S_n$), we need to prove the Submodule Theorem. First prove the Sign Lemma and deduce two corollaries [3, 2.4.1-2.4.3]; then state and prove the Submodule Theorem and, in particular, explain why this implies the Specht modules are irreducible. Finally, prove [3, Proposition 2.4.5] and from this deduce the main result of the seminar [3, Theorem 2.4.6].
**References.** The primary reference is [3, Sections 2.3 & 2.4].

**Talk 13: TBA.**

## References


1. K. Conrad, *Tensor products*, available at http://www.math.uconn.edu/ kconrad/blurbs/linmultialg/tensorprod.pdf.
2. G. D. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of mathematics and its applications, vol. 16, CUP, 2009.
3. B. E. Sagan, *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, second ed., Graduate texts in mathematics, vol. 203, Springer, 1991.
4. J.-P. Serre, *Linear representations of finite groups*, english translation by L. L. Scott of second ed., Graduate texts in mathematics, vol. 402, Springer-Verlag, 1997.
5. P. Weber, *A course in finite group representation theory*, Cambridge Studies in Advanced Mathematics, vol. 161, CUP, 2016.


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