SEMINAR ON MOTIVES AND MODULI OF BUNDLES

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Our seminar takes place on Wednesday afternoons, 16:15 - 17:45 in Arnimallee 6, SR 025/026. The first seminar will take place on the 15th October 2014. Everyone is welcome to attend and please let me know if you would like to give a talk.

Introduction

The theory of motives is envisioned to be a universal cohomology theory that will unify the various known cohomology theories (e.g. Betti, de Rham, étale) for algebraic varieties. The idea, which goes back to Grothendieck, is that the motive of an algebraic variety X should be a linearised version of X which underlies the structure of $H(X)$ for all cohomology theories. More precisely, one expects an abelian category $\mathbf{MM}(k)$ of mixed motives over k with a functor from the category of k-varieties to the category of graded mixed motives, such that mixed Weil cohomology theories on the category of k-varieties factor through this functor.

Furthermore, motives should also capture (co)homological invariants of algebraic varieties of a more algebro-geometric flavour, such as Chow groups (which can be viewed as an algebro-geometric analogues of the singular cohomology of the underlying topological space, in the sense that the elements are built from subvarieties and the product structure comes from intersecting varieties). These type of invariants are expected to be recovered using morphisms and extensions between the motive of the variety and simpler motives.

To date, much of the theory is still conjectural; however, the existence of a category of pure motives and a triangulated category of mixed motives is established. The motives of smooth projective varieties are known as pure motives and one can construct a category of pure motives $\mathbf{M}_\sim(k)$ from the category of correspondences, given a choice of equivalence relation $\sim$ on cycles. The category $\mathbf{M}_{\mathrm{hom}}(k)$ of homological pure motives satisfies a universal property for Weil cohomology theories on the category $\mathbf{SmProj}_k$. The category of pure motives should be the semi-simple part of the conjecturally category of mixed motives, which are the motives of all k-varieties. So far, the category of mixed motives has remained elusive. Following a suggestion of Beilinson and Deligne, that it may be easier to construct a triangulated category which plays the role of the derived category of $\mathbf{MM}(k)$, Voevodsky constructed a triangulated category $\mathbf{DM}_{\mathrm{gm}}(k)$ of geometric motives which conjecturally has a motivic t-structure whose heart is the category $\mathbf{MM}(k)$.

We have two main goals for our seminar.

(1) Understand the construction and properties of the category of pure motives $\mathbf{M}_\sim(k)$ and the triangulated category of mixed motives $\mathbf{DM}(k)$.
(2) Describe (co)homological invariants for moduli of bundles.

Description of the talks

Talk 1: Algebraic cycles and Roitman's theorem - Alejandra Rincón. We start by giving some generalities on algebraic cycles and intersection theory. We describe some operations on cycles: Cartesian product, push-forward, intersection product, pull-back, operations defined using correspondences (an important example is the graph of a morphism of varieties). We give the notion of an adequate equivalence relation on algebraic cycles and discuss some examples. We define the Chow ring $\mathrm{CH}(X)$ of an algebraic variety using rational equivalence on cycles and state two important results for the Chow ring: the homotopy and excision properties.

For a smooth projective complex variety X over an algebraically closed field $k \hookrightarrow \mathbb{C}$, we define the cycle map $\gamma : \mathrm{CH}^i(X) \to H^{2i}(X_{\mathrm{an}}, \mathbb{Z})$ and the Abel–Jacobi map from the Chow group of cycles of degree 0 to the Albanese variety. We explicitly describe the monodromy representation for Lefschetz pencils as a tool to study 0-cycles of fibered surface. We give a brief outline of the most relevant results on the Chow rings of K3 surfaces. We state and prove Roitman's theorem in the case of K3 surfaces that says that the Abel–Jacobi map is an isomorphism on torsion subgroups. If time permits, we give an alternative approach for elliptic K3 surfaces.

Talk 2 : An introduction to motives - Victoria Hoskins. We start with the definition of a (mixed) Weil cohomology theory in order to state the main conjecture concerning motives (universality of $\mathbf{MM}(k)$ with respect to all mixed Weil cohomology theories).

The remainder of the talk concerns pure motives, the motives of smooth projective k-varieties. The category of pure motives $\mathbf{M}_\sim(k)$ depends on a choice of adequate equivalence relation $\sim$ on cycles. The finest choice, rational equivalence, yields the category of pure Chow motives. Another important choice for us, is homological equivalence with respect to a Weil cohomology theory; it is a long-standing open conjecture that homological equivalence agrees with numerical equivalence (this is one of the "standard conjectures"). To construct $\mathbf{M}_\sim(k)$, our starting point, rather than the category of smooth projective k-varieties, is the linearisation of this category, the category of correspondences $\mathbf{Corr}_\sim(k)$ between smooth projective varieties; the advantage of allowing correspondences as morphisms is that we obtain an additive category. The category of effective pure motives $\mathbf{M}^{\mathrm{eff}}_\sim(k)$ is constructed as the pseudo-abelian envelope of $\mathbf{Corr}_\sim(k)$. This enables us to obtain direct sum decompositions associated to idempotents and so we can think of effective motives as direct factors of smooth projective varieties. The category of effective pure motives is a tensor category with unit $\mathbb{1}$ equal to the motive of the point $\operatorname{Spec} k$.

We will see that the motive of $\mathbb{P}^1$ decomposes as $\mathbb{1} \oplus \mathbb{L}$, where $\mathbb{L}$ is called the Lefschetz motive. We obtain the category of pure motives $\mathbf{M}_\sim(k)$ by formally inverting tensoring with $\mathbb{L}$ (this process rigidifies the category of effective pure motives, enabling us to take duals). We show that the category $\mathbf{M}_{\mathrm{hom}}(k)$ of pure homological motives has the universal property with respect to Weil cohomology theories on $\mathbf{SmProj}_k$. To end, we explain an idea of Beilinson and Deligne, that it may be easier to construct a triangulated category of motives that should be the derived category of the category of mixed motives.
References [1] §1-5 and [15] I.

Talk 3: Stacks and virtual motives. The first part of this talk is a quick overview of stacks with a strong focus on examples (for us, the key examples are quotient stacks $[X/G]$ and the stack of principal bundles on a curve). Algebraic stacks should be viewed as generalisations of schemes and the starting point for any discussion about stacks is usually to describe the functor of points of a variety and moduli functors.

The second part of this talk concerns virtual motives of varieties and stacks. We define the Grothendieck ring $K_0(\mathbf{Var}_k)$ of varieties and refer to the class $\mu(X)$ of a variety X as the virtual motive of X. We note that virtual motivic decompositions are much weaker than motivic decompositions, as for any open subset $U \subset X$ with closed complement Z, we have

$$\mu(X) = \mu(U) + \mu(Z);$$

in general, this decomposition is not the case for motives.

It will be convenient to be able to invert certain motivic classes, in order to define virtual motives of stacks. Therefore, we consider the dimensional completion $\widehat{K_0}(\mathbf{Var})$, which is a completion of the localisation $K_0(\mathbf{Var}_k)[\mathbb{L}^{-1}]$, where $\mathbb{L} = \mu(\mathbb{A}^1)$. Using the Bruhat decomposition of a connected split semisimple group G, we compute the virtual motive of G, and see that it is invertible in this dimensional completion. Finally, we define virtual motives of some stacks (for example, quotient stacks of the form $[X/G]$ and, more generally, stacks essentially of finite type with linear stabilisers cf. [6] Lemma 2.3).
References for stacks, [13]; for virtual motives, [6] and [7].

Talk 4: The virtual motive of the stack of principal bundles. In this talk, we provide a summary of a paper of Behrend and Dhillon on the virtual motive of the stack of principal bundles on a curve. Let C be a smooth projective connected curve over k and G a connected split semisimple algebraic group. Then we denote by $\mathfrak{Bun}_{G,C}$ the stack of principal G-bundles on C. In [6], Behrend and Dhillon give a conjectural formula for the virtual motive $\mu(\mathfrak{Bun}_{G,C})$ in terms of the virtual motive $\mathbb{L}$ of the affine line and the motivic Zeta function of C (which involves the virtual motives of symmetric powers of C).

We outline the proof of this formula given by Behrend and Dhillon in two cases. First, when $G = \mathrm{SL}_n$, we prove the formula by using an ind-variety of matrix divisors as in [8]. Second, when $C = \mathbb{P}^1$, we prove the formula using the explicit classification of G-torsors over $\mathbb{P}^1$ due to Grothendieck and Harder.
References [6].

Talk 5: Triangulated categories and Verdier localisation. In this talk, we give notion of a triangulated category. The key example for us is the homotopy category $\mathbf{K}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$, whose objects are cochain complexes of objects in $\mathcal{A}$ and whose morphisms are homotopy classes of chain maps. In this example, the auto-equivalence [1] given by shifting the complex and the distinguished triangles are triangles isomorphic to mapping cone triangles.

We give the definition of a thick subcategory $\mathcal{T}'$ of a triangulated category $\mathcal{T}$ and the notion of Verdier localisation of $\mathcal{T}$ by $\mathcal{T}'$ (in this process, one formally inverts all morphisms in the thick subcategory). The key example of this process is the construction of the derived category $\mathbf{D}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ as the localisation of $\mathbf{K}^*(\mathcal{A})$ by the class of quasi-isomorphisms. It can be difficult to describe morphism groups in the derived category (or in any Verdier localisation); although, we will see that morphisms can be described using 'roof diagrams'.

Finally, we give the notion of a t-structure on a triangulated category and its associated heart. The key example is the standard t-structure on the derived category $\mathbf{D}^b(\mathcal{A})$ whose corresponding heart is the original abelian category $\mathcal{A}$.
References [12].

Talk 6: Geometric motives via Verdier localisation. We are now in a position to describe the triangulated category $\mathbf{DM}_{\mathrm{gm}}(k)$ of geometric motives over k.

We first construct the category of effective geometric motives as follows. We define the category $\mathbf{Corr}_{\mathrm{fin}}(k)$ of finite correspondence of smooth schemes over k (note that we are now working with $\mathbf{Sm}_k$ rather than $\mathbf{SmProj}_k$). We consider the bounded homotopy category $K^b(\mathbf{Corr}_{\mathrm{fin}}(k))$ and Verdier localise a class of morphisms (which includes homotopy and Mayer-Vietoris maps). Then we take the pseudo-abelian hull to obtain the triangulated category $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ of effective geometric motives. Then we have a covariant motives functor $M^{\mathrm{eff}}_{\mathrm{gm}} : \mathbf{Sm}_k \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$.

The category of geometric motives can be viewed as a stabilisation of the category of effective geometric motives in which the Lefschetz motive is invertible for the tensor product. Hence, we can naively think of geometric motives as pairs (M, m) consisting of an effective geometric motive M and an integer m. There is a functor

$$i : \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathbf{DM}_{\mathrm{gm}}(k)$$

given by $M \mapsto (M, 0)$ which is, by Voevodsky's cancellation theorem, a fully faithful embedding.

We can also define triangulated categories $\mathbf{DM}^{(\mathrm{eff})}_{\mathrm{gm}}(k; \Lambda)$ of (effective) geometric motives over k with coefficients in a ring Λ (often $\Lambda = \mathbb{Q}$); the idea is to replace $\mathbf{Corr}_{\mathrm{fin}}(k)$ in the above construction with the category $\mathbf{Corr}_{\mathrm{fin}}(k; \Lambda)$ of finite correspondences with values in Λ.
References [15] II §3 and [1] §15-16.

Talk 7: Properties of geometric motives. In this talk, we describe without proof some important properties of motives. We define the motivic cohomology groups using Tate twists and homomorphism groups in $\mathbf{DM}_{\mathrm{gm}}(k)$. The Chow groups can be obtained as motivic cohomology groups and a result of Voevodsky states there is a full embedding $\mathbf{M}_{\mathrm{rat}}(k)^{\mathrm{op}} \hookrightarrow \mathbf{DM}_{\mathrm{gm}}(k)$. We give two important properties of motivic cohomology: homotopy invariance and the Mayer-Vietoris exact sequence. This enables us to give an explicit formula for the motive of $\mathbb{P}^n$. This formula generalises to the projective bundle formula for the motive of the projectivisation of a vector bundle on a smooth variety.

To relate back to the earlier talk on virtual motives, we describe the motive functor with compact support

$$M^c_{\mathrm{gm}} : \mathbf{Var}_k \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k;\mathbb{Q})$$

and see that this descends to a ring homomorphism

$$M^c_{\mathrm{gm}} : K_0(\mathbf{Var}_k) \to K_0(\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k;\mathbb{Q})).$$

We also mention some results concerning realisations. For $k \hookrightarrow \mathbb{C}$, there is a 'derived category $\mathbf{D}^b_{\mathrm{MR}}(k)$ of mixed realisations' and a triangulated realisation functor $\mathbf{DM}_{\mathrm{gm}}(k) \to \mathbf{D}^b_{\mathrm{MR}}(k)$ which induces a transformation of functors from motivic cohomology to absolute cohomology of mixed realisations [14]. As a consequence, there are Betti and Hodge realisation functors.

The proofs of the above results take place in a triangulated category $\mathbf{DM}(k)$ of mixed motives that contains $\mathbf{DM}_{\mathrm{gm}}(k)$. Using this larger category, we can compute morphism groups in $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ and define motivic cohomology with support; this enables one to prove the existence of a Gysin distinguished triangle in $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ associated to a closed immersion of smooth varieties. This larger category is the topic of the following talk.
References [15] II §4, [16] and [1] §17-18 and §22 for realisations.

Talk 8: Larger categories of motives. We construct triangulated categories $\mathbf{DM}(k)$ and $\mathbf{DA}(k)$ as Verdier localisations of the derived categories of Nisnevich sheaves with transfer and without transfer respectively.

We define the category $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}(k))$ (resp. $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Corr}_{\mathrm{fin}}(k))$) of Nisnevich sheaves (resp. Nisnevich sheaves with transfer) on $\mathbf{Sm}(k)$. The terminology 'with transfer' is explained by the fact that a sheaf with transfer is a sheaf F of abelian groups on $\mathbf{Sm}_k$ with transfer maps $\mathrm{Tr}(a) : F(Y) \to F(X)$ for every finite correspondence a from X to Y (with some compatibility conditions).

We define the effective category $\mathbf{DA}^{\mathrm{eff}}(k)$ (resp. $\mathbf{DM}^{\mathrm{eff}}(k)$) to be the Verdier localisation of the derived category of $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}(k))$ (resp. $\mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Corr}_{\mathrm{fin}}(k))$) by a class of $\mathbb{A}^1$-weak equivalences. The non-effective categories $\mathbf{DA}(k)$ and $\mathbf{DM}(k)$ should be constructed so that the Lefschetz motive is inverted with respect to the tensor structure (strictly speaking, one should work with a category of spectra to correctly invert the Lefschetz motive, analogously to the construction of the stable homotopy category in algebraic topology).

The forgetful functor

$$o_{\mathrm{tr}} : \mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Corr}_{\mathrm{fin}}(k)) \to \mathbf{Sh}_{\mathrm{Nis}}(\mathbf{Sm}(k))$$

has a left adjoint a_{tr} which can be derived to obtain adjunctions

$$La_{\mathrm{tr}} : \mathbf{DA}^{(\mathrm{eff})}(k;\Lambda) \rightleftarrows \mathbf{DM}^{(\mathrm{eff})}(k;\Lambda) : Ro_{\mathrm{tr}}$$

between the (effective) triangulated categories of mixed motives. In both the effective and non-effective case, this is an equivalence for $k = \mathbb{C}$ and $\Lambda = \mathbb{Q}$ (cf. [3]).

For k of characteristic zero, the comparison theorem states there is a full dense embedding

$$i : \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k) \to \mathbf{DM}^{\mathrm{eff}}_{-}(k).$$

Finally, we state Voevodsky's main theorem on $\mathbb{A}^1$-localisation for $\mathbf{DM}^{\mathrm{eff}}(k)$: any homotopy invariant (pre)sheaf with transfers is $\mathbb{A}^1$-local.
References [4], [16] and [1] §22 (where the category of bounded above complexes is used).

Possible future talks. Here are a few topics we may also be interested in studying.

(1) The motive of a reductive group [9].
(2) The motive of a stack [11].
(3) Motivic decompositions for BB decompositions [10].
(4) The motivic cohomology of the moduli space of vector bundles [2].
(5) The motivic proof of Roitman's theorem [5].
(6) Proofs of some of the results described in Talk 7 (for example, see [15]).

References

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