

Solution to Exercise sheet 6, Question 2

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Let G be an affine alg. gp acting on an irreducible affine variety X .

Assume: $\text{char}(k) = 0$ (this is important - see below).

Suppose there is a G -inv. morphism $\varphi: X \rightarrow Y$ such that

(i) $\text{codim}(Y - \varphi(X), Y) \geq 2,$

(ii) $\exists U \subseteq Y$ s.t. $\forall y \in U, \varphi^{-1}(y)$ contains a ! closed orbit,
 φ^* open

(iii) Y is a normal irreducible affine variety.

Prove that $\mathcal{O}(Y) = \mathcal{O}(X)^G$.

Solution: By (i), φ is dominant and so $\varphi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is injective.

By (ii), we have for $x_1, x_2 \in \varphi^{-1}(U)$

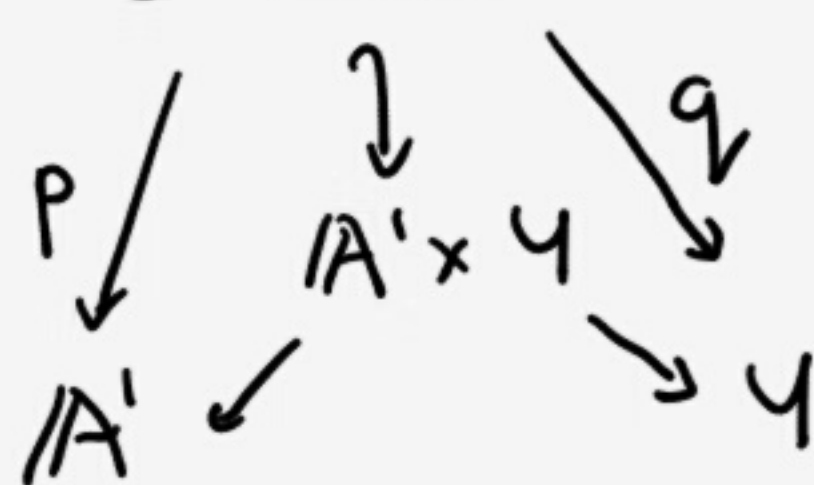
$$\varphi(x_1) = \varphi(x_2) \Rightarrow f(x_1) = f(x_2) \quad (*)$$

(as $\varphi^{-1}(\varphi(x))$ contains a ! closed orbit $G \cdot x_0 \in \overline{G \cdot x_i}$).

Claim $\forall f \in \mathcal{O}(X)^G, \exists$ rational lift $\tilde{f} \in k(Y)$ such that $\varphi^* \tilde{f} = f$.

[Pf. of claim: As $U \subseteq Y$ is a dense open, $k(U) \hookrightarrow k(Y)$ and so we may assume $Y = U$ to prove the claim.

Consider $\ell: X \rightarrow \mathbb{A}^1 \times Y$ and let $L = \overline{\ell(X)}$
 $x \mapsto (f(x), \varphi(x))$



By construction q is surjective and by (*) q is also injective.

In characteristic zero, any injective dominant morphism of irreducible varieties is birational (for example, see Humphrey's Linear Alg Gps 4.6).

Hence, there is a rational inverse to q ,

$$h: Y \dashrightarrow L.$$

↑ Then $\tilde{f} := p \circ h: Y \dashrightarrow L \rightarrow \mathbb{A}^1$ is the required lift. □

Note: the characteristic zero assumption is needed, as in char. $p > 0$,

the map $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ is bijective, but the inverse is not rational.
 $x \mapsto x^p$

We apply the result mentioned in the hint: for $\psi: M \rightarrow N$ a dominant morphism of irreducible affine varieties & $h \in k(N)$ s.t. $\psi^* h \in \mathcal{O}(M)$, h is defined on any normal point of $\psi(M)^\circ$.

Apply this to φ and \tilde{f} : as Y is normal \tilde{f} is defined on $\varphi(X)^\circ$.

By (i) $\text{codim}(Y - \varphi(X), Y) \geq 2$ and as Y is normal any function extends over a codim 2 gap (alg. Hartog's Thm). Hence $\tilde{f} \in \mathcal{O}(Y)$. □