Solution to Exercise Sheet 6, Question 2

V. Hoskins

Let G be an affine alg. gp acting on an irreducible affine variety X.

Assume: $\mathrm{char}(k) = 0$ (this is important - see below).

Suppose there is a G-inv. morphism $\varphi : X \to Y$ such that

(i) $\mathrm{codim}\,(Y - \varphi(X), Y) \geq 2$,

(ii) $\underset{\phi^*}{\exists}\ \underset{\text{open}}{U \subseteq Y}$ s.t $\forall\ y \in U$, $\varphi^{-1}(y)$ contains a ! closed orbit,

(iii) Y is a normal irreducible affine variety.

Prove that $\mathcal{O}(Y) = \mathcal{O}(X)^G$.

Solution: By (i), φ is dominant and so $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective.

By ii), we have for $x_1, x_2 \in \varphi^{-1}(U)$

$$\varphi(x_1) = \varphi(x_2) \quad \Rightarrow \quad f(x_1) = f(x_2) \qquad (*)$$

(as $\varphi^{-1}(\varphi(x))$ contains a ! closed orbit $G\cdot x_0 \in \overline{G \cdot x_i}$).

Claim $\forall\ f \in \mathcal{O}(X)^G$, $\exists$ rational lift $\tilde{f} \in k(Y)$ such that $\varphi^* \tilde{f} = f$.

[Pf. of claim: As $U \subseteq Y$ is a dense open, $k(U) \hookrightarrow k(Y)$ and so we may assume $Y = U$ to prove the claim.

Consider $l : X \to \mathbb{A}^1 \times Y$ and let $L = \overline{l(X)}$

$x \mapsto (f(x), \varphi(x))$

$$\begin{tikzcd} & L \arrow[dl, "p"'] \arrow[d, hook] \arrow[dr, "q"] & \\ \mathbb{A}^1 & \mathbb{A}^1 \times Y \arrow[l] \arrow[r] & Y \end{tikzcd}$$

By construction q is surjective and by $(*)$ q is also injective.

In characteristic zero, any injective dominant morphism of irreducible varieties is birational (for example, see Humphrey's Linear Alg Gps 4.6).

Hence, there is a rational inverse to q

$$h : Y \dashrightarrow L.$$

] Then $\tilde{f} := p \circ h : Y \dashrightarrow L \to \mathbb{A}^1$ is the required lift. $\square$

Note: the characteristic zero assumption is needed, as in char. $p > 0$, the map $\mathbb{A}^1 \to \mathbb{A}^1$, $x \mapsto x^p$ is bijective, but the inverse is not rational.

We apply the result mentioned in the hint: for $\psi : M \to N$ a dominant morphism of irreducible affine varieties & $h \in k(N)$ s.t $\psi^* h \in \mathcal{O}(M)$, h is defined on any normal point of $\psi(M)^\circ$.

Apply this to φ and $\tilde{f}$: as Y is normal $\tilde{f}$ is defined on $\varphi(X)^\circ$.

By i) $\mathrm{codim}(Y - \varphi(X), Y) \geq 2$ and a Y is normal any function extends over a codim 2 gap (alg. Hartog's Thm). Hence $\tilde{f} \in \mathcal{O}(Y)$. $\square$