Algebra III (Algebraic Geometry II) WS15-16
Fachbereich Mathematik und Informatik
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Algebraic Geometry II

Exercise Sheet 3

Hand-in date: 10am, Monday 2nd November.

Exercise 1. Consider the action of $\mathrm{GL}_n \times \mathrm{GL}_m$ on $\mathrm{Mat}_{n\times m}$ by

$$(g,h)\cdot M = gMh^{-1}.$$

Classify the orbits of this action and determine the open and closed orbits, as well as the closure of each orbit.

Exercise 2. Let G be an algebraic group scheme (over k).

i) Prove that G is separated. In particular, G is a group variety if and only if G is reduced.
ii) Prove that G is reduced if and only if G is smooth.
iii) Prove that the reduced scheme G_{red} underlying G is a smooth subgroup of G.
iv) Prove that if G is connected, then G is irreducible.

Exercise 3. Let $n > 1$ and consider the moduli functor $\mathcal{E}nd_n$ classifying endomorphisms of an n-dimensional space.

i) Show that there is a natural transformation $\eta : \mathcal{E}nd_n \to h_{\mathbb{A}^n}$ such that $\eta_{\operatorname{Spec} k}$ sends an endomorphism $T : V \to V$ of an n-dimensional vector space V to the coefficients of the characteristic polynomial of T.
   [Hint: For a family $(\mathcal{E}, T)$ over S, choose a covering S_i of S on which $\mathcal{E}$ is trivial and a basis of generating sections of $\mathcal{E}|_{S_i}$, then interpret T as a $n\times n$-matrix with coefficients in $\mathcal{O}(S_i)$.]
ii) Define a new moduli functor $\mathcal{E}nd_n^{ss}$ classifying semisimple endomorphisms of an n-dimensional space by

   $$\mathcal{E}nd_n^{ss}(S) := \{(\mathcal{E}, T) \text{ family of endomorphisms over } S : T_s \text{ is semisimple for all } s\}/\sim_S .$$

   Prove that the restriction of the above natural transformation to $\mathcal{E}nd_n^{ss}$ is a bijection at the level of k-points.

iii) Show that $\mathbb{A}^n$ is a coarse moduli space for $\mathcal{E}nd_n^{ss}$.
[Hint: To show universality with respect to all natural transformations $\mathcal{E}nd_n^{ss} \to h_N$, it may help to first assume N is affine and then cover N by affines.]

iv) Is $\mathbb{A}^n$ a fine moduli space for $\mathcal{E}nd_n^{ss}$?