## 8. Moduli of vector bundles on a curve

In this section, we describe the construction of the moduli space of (semi)stable vector bundles on a smooth projective curve $X$ (always assumed to be connected) using geometric invariant theory.

The outline of the construction follows the general method described in §2.6. First of all, we fix the available discrete invariants, namely the rank $n$ and degree $d$. This gives a moduli problem $\mathcal{M}(n,d)$, which is unbounded by Example 2.22. We can overcome this unboundedness problem by restricting to moduli of semistable vector bundles and get a new moduli problem $\mathcal{M}^{ss}(n,d)$. This moduli problem has a family with the local universal property over a scheme $R$. Moreover, we show there is a reductive group $G$ acting on $R$ such that two points lie in the same orbits if and only if they correspond to isomorphic bundles. Then the moduli space is constructed as a GIT quotient of the $G$-action on $R$. In fact, the notion of semistability for vector bundles was introduced by David Mumford following his study of semistability in geometric invariant theory, and we will see both concepts are closely related.

The construction of the moduli space of stable vector bundles on a curve was given by Seshadri [37], and later Newstead in [30, 31]. In these notes, we will essentially follow the construction due to Simpson [39] which generalises the curve case to a higher dimensional projective scheme. An in-depth treatment of the general construction following Simpson can be found in the book of Huybrechts and Lehn [16]. However, we will exploit some features of the curve case to simplify the situation; for example, we directly show that the family of semistable vector bundles with fixed invariants over a smooth projective curve is bounded, without using the Le Potier-Simpson estimates which are used to show boundedness in higher dimensions.

**Convention:** Throughout this section, $X$ denotes a connected smooth projective curve. By 'sheaf' on a scheme $Y$, we always mean a coherent sheaf of $\mathcal{O}_Y$-modules.

8.1. **An overview of sheaf cohomology.** We briefly recall the definition of the cohomology groups of a sheaf $\mathcal{F}$ over $X$. By definition, the sheaf cohomology groups $H^i(X,\mathcal{F})$ are obtained by taking the right derived functors of the left exact global sections functor $\Gamma(X,-)$. Therefore,

$$H^0(X,\mathcal{F}) \cong \Gamma(X,\mathcal{F}).$$

As $X$ is projective, $H^i(X,\mathcal{F})$ are finite dimensional $k$-vector spaces and, as $X$ has dimension 1, we have $H^i(X,\mathcal{F}) = 0$ for $i > 1$. The cohomology groups can be calculated using Cech cohomology. The first Cech cohomology group is the group of 1-cochains modulo the group of 1-coboundaries. More precisely, given a cover $\mathcal{U} = \{U_i\}$ of $X$, we let $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$ denote the double and triple intersections; then we define

$$H^1(\mathcal{U},\mathcal{F}) := Z^1(\mathcal{U},\mathcal{F})/B^1(\mathcal{U},\mathcal{F})$$

where

$$Z^1(\mathcal{U},\mathcal{F}) := \mathrm{Ker}\,\delta_1 = \{(f_{ij}) \in \bigoplus_{i,j} H^0(\mathcal{U}_{ij},\mathcal{F}) : \forall i,j,k, f_{ij} - f_{jk} + f_{ki} = 0 \in \mathcal{F}(\mathcal{U}_{ijk})\}$$

$$B^1(\mathcal{U},\mathcal{F}) := \mathrm{Image}\,\delta_0 = \{(h_i - h_j) \text{ for } (h_i) \in \bigoplus_i \mathcal{F}(U_i)\}$$

are the group of 1-cochains and 1-coboundaries respectively. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is an induced homomorphism $H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{V},\mathcal{F})$ and the first cohomology group $H^1(X,\mathcal{F})$ is the direct limit of the groups $H^1(\mathcal{U},\mathcal{F})$ over all covers $\mathcal{U}$ of $X$. In fact, these definitions of Cech cohomology groups make sense for any scheme $X$ and any coherent sheaf $\mathcal{F}$; however, higher dimensional $X$, will in general have non-zero higher degree cohomology groups.

The above definition does not seem useful for computational purposes, but it is because of the following vanishing theorem of Serre.

**Theorem 8.1** ([14] III Theorem 3.7)**.** *Let $Y$ be an affine scheme and $\mathcal{F}$ be a coherent sheaf on $Y$; then for all $i > 0$, we have*

$$H^i(Y,\mathcal{F}) = 0.$$

Consequently, we can calculate cohomology of coherent sheaves on a separated scheme using an affine open cover.

**Theorem 8.2** ([14] III Theorem 4.5)**.** *Let $Y$ be a separated scheme and $\mathcal{U}$ be an open affine cover of $Y$. Then for any coherent sheaf $\mathcal{F}$ on $Y$ and any $i \geq 0$, the natural homomorphism*

$$H^i(\mathcal{U}, \mathcal{F}) \to H^i(Y, \mathcal{F})$$

*is an isomorphism.*

The assumption that $Y$ is separated is used to ensure that the intersection of two open affine subsets is also affine (see [14] II Exercise 4.3). Hence, we can apply the above Serre vanishing theorem to all multi-intersections of the open affine subsets in the cover $\mathcal{U}$.

**Exercise 8.3.** Using the above theorem, calculate the sheaf cohomology groups

$$H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$$

by taking the standard affine cover of $\mathbb{P}^1$ consisting of two open sets isomorphic to $\mathbb{A}^1$.

One of the main reasons for introducing sheaf cohomology is that short exact sequences of coherent sheaves give long exact sequences in cohomology. The category of coherent sheaves on $X$ is an abelian category, where a sequence of sheaves is exact if it is exact at every stalk. Furthermore, a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

induces a long exact sequence in sheaf cohomology

$$0 \to H^0(X, \mathcal{E}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to 0,$$

which terminates at this point as $\dim X = 1$.

**Definition 8.4.** For a coherent sheaf $\mathcal{F}$ on $X$, we let $h^i(X, \mathcal{F}) = \dim H^i(X, \mathcal{F})$ as a $k$-vector space. Then we define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}).$$

In particular, the Euler characteristic is additive on short exact sequences:

$$\chi(\mathcal{F}) = \chi(\mathcal{E}) + \chi(\mathcal{G}).$$

8.2. **Line bundles and divisors on curves.**

**Example 8.5.**

(1) For $x \in X$, we let $\mathcal{O}_X(-x)$ denote the sheaf of functions vanishing at $x$; that is, for $U \subset X$, we have

$$\mathcal{O}_X(-x)(U) = \{f \in \mathcal{O}_X(U) : f(x) = 0\}.$$

By construction, this is a subsheaf of $\mathcal{O}_X$ and, in fact, $\mathcal{O}_X(-x)$ is an invertible sheaf on $X$.

(2) For $x \in X$, we let $k_x$ denote the skyscraper sheaf of $x$ whose sections over $U \subset X$ are given by

$$k_x(U) := \begin{cases} k & \text{if } x \in U \\ 0 & \text{else.} \end{cases}$$

The skyscraper sheaf is not a locally free sheaf; it is a torsion sheaf which is supported on the point $x$. Since $H^0(X, k_x) = k_x(X) = k$ and $H^1(X, k_x) = 0$, we have $\chi(k_x) = 1$.

There is a short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to k_x \to 0 \qquad (2)$$

where for $U \subset X$, the homomorphism $\mathcal{O}_X(U) \to k_x(U)$ is given by evaluating a function $f \in \mathcal{O}_X(U)$ at $x$ if $x \in U$. We can tensor this exact sequence by an invertible sheaf $\mathcal{L}$ to obtain

$$0 \to \mathcal{L}(-x) \to \mathcal{L} \to k_x \to 0$$

where $\mathcal{L}(-x)$ is also an invertible sheaf, whose sections over $U \subset X$ are the sections of $\mathcal{L}$ over $U$ which vanish at $x$. Hence, we have the following formula

$$\chi(\mathcal{L}) = \chi(\mathcal{L}(-x)) + 1. \tag{3}$$

**Definition 8.6.** Let $X$ be a smooth projective curve.

(i) A *Weil divisor* on $X$ is a finite formal sum of points $D = \sum_{x \in X} m_x x$, for $m_x \in \mathbb{Z}$.
(ii) The *degree* of $D$ is $\deg D = \sum m_x$.
(iii) We say $D$ is *effective*, denoted $D \geq 0$, if $m_x \geq 0$ for all $x$.
(iv) For a rational function $f \in k(X)$, we define the associated *principal divisor*

$$\mathrm{div}(f) = \sum_{x \in X(k)} \mathrm{ord}_x(f)x,$$

where $\mathrm{ord}_x(f)$ is the order of vanishing of $f$ at $x$ (as $\mathcal{O}_{X,x}$ is a discrete valuation ring, we have a valuation $\mathrm{ord}_x : k(X)^* \to \mathbb{Z}$).
(v) We say two divisors are *linearly equivalent* if their difference is a principal divisor.
(vi) For a Weil divisor $D$, we define an invertible sheaf $\mathcal{O}_X(D)$ by

$$\mathcal{O}_X(D)(U) := \{0\} \cup \{f \in k(X)^* : (\mathrm{div} f + D)|_U \geq 0\}.$$

**Remark 8.7.**

(1) For $D = -x$, this definition of $\mathcal{O}_X(D)$ coincides with the definition of $\mathcal{O}_X(-x)$ above.
(2) As $X$ is smooth, the notions of Weil and Cartier divisors coincide. The above construction $D \mapsto \mathcal{O}_X(D)$ determines a homomorphism from the group of Weil divisors modulo linear equivalence to the Picard group of isomorphism classes of line bundles, and this homomorphism is an isomorphism as $X$ is smooth. In particular, any invertible sheaf $\mathcal{L}$ over $X$ is isomorphic to an invertible sheaf $\mathcal{O}_X(D)$. For proofs of these statements, see [14] II §6.

For an effective divisor $D$, the dual line bundle $\mathcal{O}(-D)$ is isomorphic to the ideal sheaf of the (possibly non-reduced) subscheme $D \subset X$ given by this effective divisor (see [14] II Proposition 6.18) and we have a short exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to k_D \to 0, \tag{4}$$

where $k_D$ denotes the skyscraper sheaf supported on $D$; thus $k_D$ is a torsion sheaf. This short exact sequence generalises the short exact sequence (2). In particular, any effective divisor admits a non-zero section $\mathcal{O}_X \to \mathcal{O}_X(D)$. In fact, a line bundle $\mathcal{O}_X(D)$ admits a non-zero section if and only if $D$ is linearly equivalent to an effective divisor $D$ by [14] II Proposition 7.7.

**Definition 8.8.** The Grothendieck group of $X$, denoted $K_0(X)$, is the free group generated by classes $[\mathcal{E}]$ ,for $\mathcal{E}$ a coherent sheaf on $X$, modulo the relations $[\mathcal{E}] - [\mathcal{F}] + [\mathcal{G}] = 0$ for short exact sequences $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.

We claim that there is a homomorphism

$$(\det, \mathrm{rk}) : K_0(X) \to \mathrm{Pic}(X) \oplus \mathbb{Z} \tag{5}$$

which sends a locally free sheaf $\mathcal{E}$ to $(\det \mathcal{E} := \wedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}, \mathrm{rk}\,\mathcal{E})$. To extend this to a homomorphism on $K_0(X)$, we need to define the map for coherent sheaves $\mathcal{F}$: for this, we can take a finite resolution of $\mathcal{F}$ by locally free sheaves, which exists because $X$ is smooth, and use the relations defining $K_0(X)$. This map is surjective and in fact is an isomorphism (see [14] II, Exercise 6.11). Using this homomorphism we can define the degree of any coherent sheaf on $X$.

**Definition 8.9.** (The *degree* of a coherent sheaf).

(i) If $D$ is a divisor, we define $\deg \mathcal{O}_X(D) := \deg D$.
(ii) If $\mathcal{F}$ is a torsion sheaf, we define $\deg \mathcal{F} = \sum_{x \in X} \mathrm{length}(\mathcal{F}_x)$.
(iii) If $\mathcal{E}$ is a locally free sheaf, $\deg \mathcal{E} = \deg(\det \mathcal{E})$.
(iv) If $\mathcal{F}$ is a coherent sheaf, we define $\deg \mathcal{E} := \deg(\det \mathcal{F})$, where $\det \mathcal{F}$ is the image of $\mathcal{F}$ in Pic(X) under the homomorphism (5).

In fact, the degree is uniquely determined by the first two properties and the fact that the degree is additive on short exact sequences (that is, if we have a short exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$, then $\deg \mathcal{F} = \deg \mathcal{E} + \deg \mathcal{G}$); see [14] II, Exercise 6.12.

**Example 8.10.** The skyscraper sheaf $k_x$ has degree 1.

8.3. **Serre duality and the Riemann-Roch Theorem.**

**Proposition 8.11** (Riemann-Roch Theorem, version I)**.** *Let $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on a smooth projective curve $X$. Then*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$$

*Proof.* We can write $D = x_1 + \cdots + x_n - y_1 - \cdots - y_m$ and then proceed by induction on $n + m \in \mathbb{Z}$. The base case where $D = 0$ is immediate. Now assume that the equality has been proved for $D$; then we can deduce the statement for $D+x$ (and $D-x$) from the equality (3). □

**Definition 8.12.** For a smooth projective curve $X$, the sheaf of differentials $\omega_X := \Omega^1_X$ on $X$ is called the *canonical sheaf*. The *genus* of $X$ is $g(X) := h^0(X, \omega_X)$.

The canonical bundle is a locally free sheaf of rank $1 = \dim X$; see [14] II Theorem 8.15.

**Theorem 8.13** (Serre duality for a curve)**.** *Let $X$ be a smooth projective curve and $\mathcal{E}$ be a locally free sheaf over $X$. There exists a natural perfect pairing*

$$H^0(X, \mathcal{E}^\vee \otimes \omega_X) \times H^1(X, \mathcal{E}) \to k.$$

*Hence, $H^0(X, \mathcal{E}^\vee \otimes \omega_X) \cong H^1(X, \mathcal{E})^\vee$ and $h^0(X, \mathcal{E}^\vee \otimes \omega_X) = h^1(X, \mathcal{E})$.*

**Remark 8.14.**

(1) Once one chooses an isomorphism $H^1(X, \omega_X) \simeq k$, the pairing can be described as the composition

$$H^0(X, \mathcal{E}^\vee \otimes \omega_X) \times H^1(X, \mathcal{E}) \to H^1(X, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega_X) \to H^1(X, \omega_X) \simeq k$$

where the first map is a cup-product and the map $\mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$ is the trace.

(2) In fact, Serre duality can be generalised to any projective scheme (see [14] III Theorem 7.6 for the proof) where $\omega_X$ is replaced by a dualising sheaf. If $Y$ is a smooth projective variety of dimension $n$, then the dualising sheaf is the canonical sheaf $\omega_Y = \wedge^n \Omega_Y$, which is the $n$th exterior power of the sheaf of differentials, and the first cohomology group is replaced by the $n$th cohomology group.

An important consequence of Serre duality on curves is the Riemann–Roch Theorem.

**Theorem 8.15** (Riemann–Roch theorem, version II)**.** *Let $X$ be a smooth projective curve of genus $g$ and let $\mathcal{L}$ be a degree $d$ invertible sheaf on $X$. Then*

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \omega_X) = d + 1 - g.$$

*Proof.* First, we use Serre duality to calculate the Euler characteristic of the structure sheaf

$$\chi(\mathcal{O}_X) := h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) = 1 - h^0(X, \omega_X) = 1 - g.$$

Then by Serre duality and the baby version of Riemann–Roch it follows that

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^\vee \otimes \omega_X) = \chi(\mathcal{L}) = d + \chi(\mathcal{O}(X)) = d + 1 - g$$

as required. □

There is a Riemann–Roch formula for locally free sheaves due to Weil. The proof is given by induction on the rank of the locally free sheaf with the above version giving the base case. To go from a given locally free sheaf $\mathcal{E}$ to a locally free sheaf of lower rank $\mathcal{E}'$ one uses a short exact sequence

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}' \to 0,$$

where $\mathcal{L}$ is an invertible subsheaf of $\mathcal{E}$ of maximal degree (this forces the quotient $\mathcal{E}'$ to be locally free; see Exercise 8.23 for the existence of such a short exact sequence).

**Corollary 8.16** (Riemann–Roch for locally free sheaves on a curve)**.** *Let $X$ be a smooth projective curve of genus $g$ and $\mathcal{F}$ be a locally free sheaf of rank $n$ and degree $d$ over $X$. Then*

$$\chi(\mathcal{F}) = d + n(1-g).$$

**Example 8.17.** On a curve $X$ of genus $g$, the canonical bundle has degree $2g-2$ by the Riemann–Roch Theorem:

$$h^0(X, \omega_X) - h^1(X, \mathcal{O}_X) = g - 1 = \deg \omega_X + 1 - g.$$

Therefore, on $\mathbb{P}^1$, we have $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$.