THE KEMPF–NESS THEOREM

VICTORIA HOSKINS

1. Introduction

In this talk, we will prove the Kempf–Ness theorem which relates certain algebraic quotients with certain symplectic quotients. More precisely, let G be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n_{\mathbb{C}}$. Then one can consider the geometric invariant theory quotient $\pi : X^{ss} \to X//G$, which is a categorical quotient of the open subset of semistable points in X. As G is complex reductive, G is equal to the complexification of its maximal compact subgroup K. Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ has a natural symplectic (in fact, Kähler) structure given by the Fubini-Study form. The restriction of this to X gives X a symplectic structure ω. Assuming that K acts unitarily, which one can always arrange by rechoosing coordinates on $\mathbb{P}^n$ if necessary, there is a moment map $\mu : X \to \mathfrak{k}^*$ for this action. One can alternatively consider the symplectic reduction $\mu^{-1}(0)/K$, which is a universal symplectic quotient if K acts freely on $\mu^{-1}(0)$.

Theorem 1.1 (the Kempf-Ness Theorem [2])**.** *There is an inclusion $\mu^{-1}(0) \subset X^{ss}$ which induces a homeomorphism between the symplectic reduction and the GIT quotient*

$$\mu^{-1}(0)/K \cong X//G.$$

There are many excellent expositions of this result; for example, [7] §8 and [10]. Before we prove this result, we start with an overview of the construction of these quotients.

2. An overview of quotients in algebraic and symplectic geometry

2.1. **Algebraic quotients.** The construction of quotients by group actions in algebraic geometry is given by geometric invariant theory (GIT). Let us recall the construction of GIT quotients over an algebraically closed field k (later on, when we compare GIT quotients, with symplectic reductions, we will assume that $k = \mathbb{C}$). As every GIT quotient is constructed by gluing affine GIT quotients, we start with the affine theory first.

Let G be a reductive group acting on an affine scheme $X = \operatorname{Spec} A$ of finite type over k; then there is an induced action of G on the finitely generated k-algebra A and the invariant ring A^G is also finitely generated as a k-algebra by Nagata's theorem. The inclusion $A^G \hookrightarrow A$ of finitely generated k-algebras induces a morphism of affine varieties

$$\pi : X \to X//G := \operatorname{Spec} A^G,$$

called the affine GIT quotient, which is a categorical (and good) quotient of the G-action on X. In general, this is not an orbit space, but there is an open set of 'stable' points on which π restricts to a geometric quotient (which, in particular, is an orbit space).

The affine GIT quotient is constructed by taking the spectrum of the subring of invariant functions in the coordinate ring of X. To construct a GIT quotient of a reductive group G acting on a (quasi-)projective scheme X of finite type over k, the basic idea is to instead take the projective spectrum of a graded ring of invariants.

If $X \subset \mathbb{P}^n$ is a projective scheme, then its homogeneous coordinate ring is a graded ring $R = \oplus_{r \geq 0} R_r$ such that $R_0 = k$ and R is a finitely generated graded k-algebra (in fact, the generators lie in R_1). We will refer to such graded rings R as graded k-algebras which are finitely generated in degree 1. However, the coordinate ring does not just depend on X: it depends on the projective embedding $X \hookrightarrow \mathbb{P}^n$. Conversely, if we take the projective spectrum of a graded k-algebra R which is finitely generated in degree 1, then the projective scheme $X := \operatorname{Spec} R$ also comes equipped with a very ample line bundle $\mathcal{O}(1)$. We will refer to a pair

(X, L) consisting of a projective scheme X with a very ample line bundle L, as a polarised projective scheme. A very ample line bundle L on X determines a projective embedding $X \hookrightarrow \mathbb{P}^n$. If $(X, L) = (\operatorname{Proj} R, \mathcal{O}(1))$, then the homogeneous coordinate ring of X for the associated projective embedding $X \hookrightarrow \mathbb{P}^n$ is R. More precisely, there is a bijective correspondence

$$\begin{array}{ccc} \{(X,L) \text{ polarised proj. scheme }\} & \longleftrightarrow & \{R \text{ graded } k\text{-algebra finitely gen. in degree } 1\} \\ & & \\ (\operatorname{Proj} R, \mathcal{O}(1)) & \leftarrow & R \\ (X,L) & \rightarrow & R(X,L) := \oplus_{r \geq 0} H^0(X, L^{\otimes r}). \end{array}$$

However, we also need the action of G on X to lift to an action of G on $R(X, L)$ for some line bundle L. This is given by the notion of a linearisation of the action.

Definition 2.1. Let G be a group acting on a quasi-projective scheme X. A linearisation of the action is a line bundle L on X with a G-action on L such that the projection $L \to X$ is equivariant and the morphisms on the fibres $L_x \to L_{g\cdot x}$ are linear. We say a linearisation is (very) ample if the associated line bundle is.

Example 2.2. If G acts on on $\mathbb{P}^n$ by a representation $G \to \mathrm{GL}_{n+1}$, then there is a natural linearisation of the action on $\mathcal{O}_{\mathbb{P}^n}(1)$. Indeed, the action lifts to the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$ and we can identify the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ with the blow up of $\mathbb{A}^{n+1}$ at the origin: the natural action on $\mathbb{A}^{n+1} \times \mathbb{P}^n$ preserves $\mathcal{O}_{\mathbb{P}^n}(-1)$ and this action on $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a linearisation of the action. Dually there is a linearisation of the G-action on $\mathcal{O}_{\mathbb{P}^n}(1)$. If $X \subset \mathbb{P}^n$ is preserved by the G-action, then there is also a linearisation on $\mathcal{O}(1)|_X$; in this case, we say that G acts linearly on $X \subset \mathbb{P}^n$.

Let G be a reductive group acting on a projective scheme X with respect to an ample line bundle L. As the action is linear, there is an induced G-action on the graded ring $R(X, L)$ of sections of powers of L, which preserves the graded pieces. The inclusion $R(X, L)^G \hookrightarrow R(X, L)$ induces a rational map

$$X = \operatorname{Proj} R(X, L) \dashrightarrow \operatorname{Proj} R(X, L)^G$$

which is undefined on the closed subscheme of X corresponding to the homogeneous ideal $R(X, L)^G_+$ in $R(X, L)$.

Definition 2.3. In the above setting, we make the following definitions.

(1) A point $x \in X$ is *semistable* (with respect to L) if there exists $r > 0$ and an invariant section $\sigma \in R(X, L^{\otimes r})^G$ such that $\sigma(x) \neq 0$. The set of semistable points in X (with respect to L) form an open subscheme $X^{ss}(L) = X - V(R(X, L)^G_+)$.

(2) The *(projective) GIT quotient of G acting on X with respect to L* is the morphism

$$\pi : X^{ss}(L) \to X//_L G := \operatorname{Proj} R(X, L)^G$$

obtained by restricting the above rational map to its domain of definition.

Theorem 2.4 (Mumford [7])**.** *Let G be a reductive group acting on a projective scheme X with respect to an ample linearisation; then the GIT quotient*

$$\pi : X^{ss}(L) \to X//_L G := \operatorname{Proj} R(X, L)^G$$

is a good quotient of the G-action on $X^{ss}(L)$.

The idea of the proof is to use the fact that a good quotient is local on the target and to prove that the above quotient is obtained by gluing affine GIT quotients, which are good quotients. More precisely, by Nagata's theorem, we can choose finitely many invariant sections $s_1, \dots, s_m$ which generate $R(X, L)^G$ as a k-algebra; then

$$X^{ss}(L) = \bigcup_{i=1}^m X_{s_i} \quad \text{and} \quad Y := X//_L G = \bigcup_{i=1}^m Y_{s_i}$$

and X_{s_i} and Y_{s_i} are affine schemes, as L is ample. Then one can show that π is obtained by gluing affine GIT quotients

$$\pi_i : X_{s_i} \to X_{s_i}//G = Y_{s_i}.$$

Remark 2.5.

(1) The projective GIT quotient $X//_L G$ is a projective scheme.
(2) As a consequence of π being a good quoient, the k-points of $X//_L G$ are in bijective correspondence with the 'polystable orbits' (that is, the orbits that are closed in $X^{ss}(L)$).
(3) The projective GIT quotient restricts to a geometric quotient on the set of 'stable points', which is an open subset of X whose k-points are points whose G-orbits are closed in $X^{ss}(L)$ and have dimension equal to the dimension of G.
(4) More generally, Mumford constructs quasi-projective GIT quotients of quasi-projective schemes with respect to a linearisation of the action [7].

Example 2.6. Consider $G = \mathbb{G}_m$ acting linearly on $X = \mathbb{P}^n$ by

$$t \cdot [x_0 : x_1 : \cdots : x_n] = [t^{-1}x_0 : tx_1 : \cdots : tx_n].$$

Then we have

$$R(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^G = \oplus_{r\geq 0} k[x_0, \ldots, x_n]_r^{\mathbb{G}_m} = k[x_0x_1, \ldots, x_0x_n].$$

The semistable locus is $(\mathbb{P}^n)^{ss} = \mathbb{P}^n - V(x_0x_1, \ldots, x_0x_n)$

$$(\mathbb{P}^n)^{ss} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_0 \neq 0 \text{ and } (x_1, \ldots, x_n) \neq 0\} \cong \mathbb{A}^n - \{0\}$$

and the GIT quotient is

$$\varphi : (\mathbb{P}^n)^{ss} \cong \mathbb{A}^n - \{0\} \to X//G = \operatorname{Proj} k[x_0x_1, \ldots, x_0x_n] \cong \mathbb{P}^{n-1}.$$

In fact, this is a geometric quotient and the stable locus coincides with the semistable locus.

In general, it is not so easy to compute the semistable points for an action. Often one uses the Hilbert–Mumford criterion, which is a numerical criterion, to determine semistability. This follows from the following topological criterion for semistability, which we will use later on in the proof of the Kempf–Ness Theorem.

Proposition 2.7 (Topological criterion for semistability)**.** *Let G be a reductive group acting linearly on $X \subset \mathbb{P}^n$. For $x \in \mathbb{P}^n$, choose a non-zero lift $\tilde{x}$ in the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$. Then the following statements hold.*

(1) *x is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$.*
(2) *x is polystable if and only if $G \cdot \tilde{x}$ is closed.*
(3) *x is stable if and only if $G \cdot \tilde{x}$ is closed and has dimension equal to the dimension of G.*

2.2. **Symplectic quotients.** The construction of quotients by group actions in symplectic geometry is given by the symplectic reduction. Let K be a Lie group acting symplectically on a symplectic manifold (X, ω). For dimension reasons alone, the topological quotient X/K may not be symplectic. Instead one uses a moment map for the action to construct a symplectic reduction of the action. The moment map is essentially a lift of the infinitesimal action $\mathfrak{k} \to \operatorname{Vect}(X)$ given by $A \mapsto A_X$, where

$$A_{X,x} = \frac{d}{dt} \exp(tA) \cdot x|_{t=0}.$$

Definition 2.8. A moment map for K acting on (X, ω) is a smooth map $\mu : X \to \mathfrak{k}^*$ such that

(1) μ is K-equivariant (where K acts on $\mathfrak{k}^*$ by the coadjoint action),
(2) μ lifts the infinitesimal action, in the sense that, for all $A \in \mathfrak{k}$, we have

$$d\mu^A = \iota_{A_X}\omega := \omega(A_X, -),$$

where $\mu^A : X \to \mathbb{R}$ is the map given by $x \mapsto \mu(x) \cdot A$.

The following example will be very useful later on.

Example 2.9. Consider the unitary group $\mathrm{U}(n+1)$ acting on complex projective space $\mathbb{P}^n_{\mathbb{C}}$ by its standard representation on $\mathbb{C}^{n+1}$. We recall that $\mathbb{P}^n_{\mathbb{C}}$ is a symplectic manifold, with the Fubini-Study form ω_{FS} constructed from the standard Hermitian inner product H on $\mathbb{C}^{n+1}$. As H is

$\mathrm{U}(n+1)$-invariant, it follows that the action preserves ω_{FS}. Let $p = (p_0, \dots, p_n) \in \mathbb{C}^{n+1} - \{0\}$; then there is a natural moment map given by

$$\mu([p]) \cdot A = \frac{\mathrm{Tr} p^* A p}{2i||p||^2}$$

where $[p] \in \mathbb{P}^n$ and $A \in \mathfrak{u}(n+1)$ and p^* denotes the complex conjugate transpose of p.

For a coadjoint fixed point $\eta \in \mathfrak{K}^*$, the equivariance of μ implies that the preimage $\mu^{-1}(\eta)$ is preserved by the action of K.

Definition 2.10. The symplectic reduction at η is the orbit space

$$X//_{\eta}^{\mathrm{red}} K := \mu^{-1}(\eta)/K.$$

Theorem 2.11 (Marsden–Weinstein–Meyer [5, 6])**.** *Let K be a Lie group acting on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{K}^*$. If $\eta \in \mathfrak{K}^*$ is fixed by the coadjoint action and the action of K on $\mu^{-1}(\eta)$ is free and proper, then the following statements hold.*

i) *The symplectic reduction $\mu^{-1}(\eta)/K$ is a smooth manifold of dimension* $\dim X - 2\dim K$.
ii) *There is a unique symplectic form ω^{red} on $\mu^{-1}(\eta)/K$ such that $\pi^*\omega^{\mathrm{red}} = i^*\omega$ where $i : \mu^{-1}(\eta) \hookrightarrow X$ is the inclusion and $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is the quotient map.*

Let us recall the idea of the proof. It follows from the infinitesimal lifting property of μ that η is a regular value of μ if and only if the action of K on the level set $\mu^{-1}(\eta)$ is locally free. By assumption, this action is free and so η is a regular value. Hence $\mu^{-1}(\eta) \subset X$ is a smooth submanifold of dimension $\dim X - \dim K$ by the Preimage Theorem. Since K acts freely and properly on $\mu^{-1}(\eta)$, the topological quotient can be given a smooth structure by the Slice Theorem such that $\pi : \mu^{-1}(\eta) \to \mu^{-1}(\eta)/K$ is a principal K-bundle. The infinitesimal lifting property of μ combined with the fact that η is a regular value of μ can then be used to prove that for all $x \in \mu^{-1}(\eta)$, we have

$$(T_x(K \cdot x))^{\omega_x} = \ker d_x\mu = T_x\mu^{-1}(\eta).$$

Therefore, $T_x(K \cdot x)$ is an isotropic subspace (that is, $T_x(K \cdot x) \subset (T_x(K \cdot x))^{\omega_x}$) and so there is an induced symplectic form on the quotient vector space

$$(T_x(K \cdot x))^{\omega_x}/T_x(K \cdot x) = T_x\mu^{-1}(\eta)/T_x(K \cdot x) \cong T_{\pi(x)}(\mu^{-1}(\eta)/K).$$

This produces the required symplectic form on $\mu^{-1}(\eta)/K$.

Example 2.12. Consider the action of $K = S^1 \cong U(1)$ on complex projective space $\mathbb{P}^n_{\mathbb{C}}$ (with its Fubini–Study form ω_{FS}) by

$$s \cdot [x_0 : \cdots : x_n] = [s^{-1}x_0 : sx_1 : \cdots : sx_n].$$

Since S^1 acts by a representation $\rho : S^1 \to \mathrm{U}(n+1)$, the moment map for this S^1-action $\mu : \mathbb{P}^n_{\mathbb{C}} \to \mathbb{R} \cong (\mathrm{Lie}S^1)^*$ is the composition of the moment map $\mu_{\mathrm{U}(n+1)} : \mathbb{P}^n_{\mathbb{C}} \to \mathfrak{u}(n+1)^*$ for the $\mathrm{U}(n+1)$-action followed by the projection $\rho^* : \mathfrak{u}(n+1) \to \mathfrak{u}(1)^*$. By Example 2.9, we have

$$\mu([x_0 : \cdots : x_n]) = \frac{-|x_0|^2 + |x_1|^2 + \cdots |x_n|^2}{\sum_{k=0}^n |x_k|^2}.$$

Then $\mu^{-1}(0) = \{[x_0 : \cdots : x_n] : |x_0|^2 = \sum_{k=1}^n |x_k|^2\}$ and so

$$\mu^{-1}(0) \cong \{(X_1, \dots, X_n) \in \mathbb{C}^n \cong (\mathbb{P}^n_{\mathbb{C}})_{x_0 \neq 0} : \sum_{i=1}^n |X_i|^2 = 1\} \cong S^{2n-1}$$

and

$$\mu^{-1}(0)/K \cong S^{2n-1}/S^1 \cong \mathbb{P}^{n-1}_{\mathbb{C}}.$$

2.3. **An example of the Kempf–Ness Theorem.** In Examples 2.6 and 2.12, we saw the first example of the Kempf–Ness Theorem. Let $G = \mathbb{G}_m = \mathbb{C}^*$ act on $\mathbb{P}^n_{\mathbb{C}}$ by the representation $\rho : \mathbb{G}_m \to \mathrm{GL}_{n+1}$

$$t \mapsto \mathrm{diag}(t^{-1}, t, \dots, t).$$

Then the GIT quotient is

$$\pi : (\mathbb{P}^n_{\mathbb{C}})^{ss} = \mathbb{P}^n_{\mathbb{C}} - V(x_0x_1, \dots, x_0x_n) \cong \mathbb{A}^n - \{0\} \to \mathbb{P}^n_{\mathbb{C}}/\!/\mathbb{G}_m = \mathbb{P}^{n-1}_{\mathbb{C}}.$$

The maximal compact subgroup $K = \mathrm{U}(1) \cong S^1$ of G acts symplectically on $(\mathbb{P}^n_{\mathbb{C}}, \omega_{FS})$ with symplectic reduction

$$\mu^{-1}(0)/K = \{[x_0 : \cdots : x_n] : |x_0|^2 = \sum_{k=1}^{n} |x_k|^2\}/K \cong S^{2n-1}/S^1 \cong \mathbb{P}^{n-1}_{\mathbb{C}}.$$

In particular, there is an inclusion $\mu^{-1}(0) \subset (\mathbb{P}^n_{\mathbb{C}})^{ss}$ which induces a homeomorphism

$$\mu^{-1}(0)/K \cong S^{2n-1}/S^1 \cong \mathbb{P}^{n-1}_{\mathbb{C}} \cong \mathbb{P}^n_{\mathbb{C}}/\!/\mathbb{G}_m.$$

3. The proof of the Kempf–Ness Theorem

3.1. **Complex reductive groups.** If K is a real compact Lie group, we recall that its complexification $G := K_{\mathbb{C}}$ is a complex Lie group which contains K and the Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{K}$ of K ($\mathfrak{g} = \mathfrak{K} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{K} \oplus i\mathfrak{K}$). We recall the following standard result about complex reductive groups (for a proof see, for example, [9] Chapter 5 §2.5).

Theorem 3.1. *The operation of complexification defines a one-to-one correspondence between isomorphism classes of compact real Lie groups and complex reductive groups.*

Example 3.2. We list a few examples of pairs (G, K) consisting of a complex reductive group G and its maximal compact torus: $((\mathbb{C}^*)^r, (S^1)^r)$, $(\mathrm{GL}_n(\mathbb{C}), \mathrm{U}(n))$ and $(\mathrm{SL}_n(\mathbb{C}), \mathrm{SU}(n))$.

3.2. **The first version of the Kempf–Ness theorem.** Let G be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ via a representation $\rho : G \to \mathrm{GL}_{n+1}(\mathbb{C})$. Let K be the maximal compact subgroup of G, so that $G = K_{\mathbb{C}}$. Then K also acts on $\mathbb{P}^n$ and, as K is compact, we can choose coordinates on $\mathbb{P}^n$ so ρ restricts to a unitary representation $\rho : K \to \mathrm{U}(n+1)$ of K; hence, the associated Fubini–Study form ω_{FS} for this choice of coordinates on $\mathbb{P}^n_{\mathbb{C}}$ is the preserved by the action of K. Then the action is Hamiltonian with moment map $\mu : X \to \mathfrak{K}^*$ given by the composition

$$X \hookrightarrow \mathbb{P}^n_{\mathbb{C}} \xrightarrow{\mu_{\mathrm{U}(n+1)}} \mathfrak{u}(n+1)^* \xrightarrow{\rho^*} \mathfrak{k}^*,$$

where $\mu_{\mathrm{U}(n+1)}$ is the moment map given in Example 2.9.

Theorem 3.3 (The first version of the Kempf–Ness theorem [2])**.** *Let $G = K_{\mathbb{C}}$ be a complex reductive group acting linearly on a smooth complex projective variety $X \subset \mathbb{P}^n$ such that its maximal compact subgroup K acts unitarily with moment map $\mu : X \to \mathfrak{K}^*$. Then the following statements hold for closed points $x \in X$.*

i) $\overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset \iff x \in X^{ss}$.
ii) $G \cdot x \cap \mu^{-1}(0) \neq \emptyset \iff x \in X^{ps} := \{x \in X^{ss} : G \cdot x \subset X^{ss} \text{ is closed }\}$.
iii) *If $G \cdot x \cap \mu^{-1}(0) \neq \emptyset$, then this intersection is a single K-orbit.*

In particular, we have a chain of inclusions $\mu^{-1}(0) \subset G \cdot \mu^{-1}(0) = X^{ps} \subset X^{ss}$.

3.3. **The proof of the first version.** For the proof of the theorem, we shall assume for simplicity that $X = \mathbb{P}^n$ as the version for X follows from the version for $\mathbb{P}^n$ (since $X^{(s)s} = X \cap (P^n)^{(s)s}$ and similarly for the polystable locus).

Let $H : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}$ be the standard $\mathrm{U}(n+1)$-invariant Hermitian inner product on $\mathbb{C}^{n+1}$. Then the Fubini–Study form ω on $\mathbb{P}^n$ is constructed via the projection $\mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ from the imaginary part of H; that is, if $v \in \mathbb{C}^{n+1} - \{0\}$ and we identify $T_{[v]}\mathbb{P}^n$ with the orthogonal space to v in $\mathbb{C}^{n+1}$, then

$$\omega_{\mathrm{FS},[v]}(u,w) = \frac{\omega_{\mathbb{C}^{n+1}}(u,w)}{||v||^2} = \frac{1}{2i||v||^2}[H(u,w) - H(w,u)].$$

Then the moment map $\mu : \mathbb{P}^n \to \mathfrak{K}^*$ is given by

$$\mu([v]) \cdot A = \frac{\mathrm{Tr}(\overline{v}^t Av)}{2i||v||^2} = \frac{1}{2}\omega_{\mathrm{FS},[v]}(Av, v) = \frac{H(Av,v)}{2i||v||^2},$$

where we view $A \in \mathfrak{k}$ as an element of $\mathfrak{u}(n+1)$ via the representation ρ.

Let $|| - ||$ denote the norm associated to the Hermitian inner product H. Then as in [2], we consider the non-negative function

$$\begin{aligned} p_v : G &\to \mathbb{R} \\ g &\mapsto ||g \cdot v||^2. \end{aligned}$$

We note that p_v is constant on K, as H is K-invariant.

Lemma 3.4. *An element $g \in G$ is a critical point of p_v if and only if $\mu(g \cdot [v]) = 0$.*

Proof. As $p_v(g) = p_{g \cdot v}(e)$, we see that g is a critical value of p_v if and only if e is a critical value of $p_{g \cdot v}$. Hence, it suffices to prove that e is a critical value of p_v if and only if $\mu([v]) = 0$. Since p_v is constant on K, it suffices to consider elements of the form $0 + iA \in \mathfrak{k} \oplus i\mathfrak{k} = \mathfrak{k}_{\mathbb{C}} = \mathfrak{g}$. Then we have

$$\begin{aligned} d_e p_v(iA) &:= \frac{d}{dt}||\exp(itA) \cdot v||^2|_{t=0} \\ &= H(\frac{d}{dt}\exp(itA) \cdot v|_{t=0}, v) + H(v, \frac{d}{dt}\exp(itA) \cdot v|_{t=0}) \\ &= H(iAv, v) + H(v, iAv) \\ &= i[H(Av,v) - H(v,Av)] \\ &\overset{(1)}{=} 2iH(Av,v) \\ &= -4||v||^2\mu([v]) \cdot A, \end{aligned}$$

where the final equality is by definition of μ and the penultimate equality relies on the fact that for $v \in \mathbb{C}^{n+1}$ and $A \in \mathfrak{u}(n+1)$ one has

$$H(Av,v) + H(v,Av) = 0. \tag{1}$$

Hence, e is a critical point of p_v if and only if $\mu([v]) = 0$. □

Lemma 3.5. *For $v \in \mathbb{C}^{n+1}$, the following statements are equivalent:*

i) *$G \cdot v$ is closed;*
ii) *p_v has a minimum;*
iii) *p_v has a critical point.*

Proof. i) $\implies$ ii): If $G \cdot v$ is closed, then also $||G \cdot v||^2$ is closed, as the norm $|| - ||^2$ is proper (as it is continuous and the preimage of a bounded set is bounded). Therefore,

$$\inf_g p_v(g) \in \overline{||G \cdot v||^2} = ||G \cdot v||^2 = \mathrm{Im}\, p_v;$$

that is, p_v attains a minimum.

ii) $\implies$ iii): as any minimum is a critical point.

iii) $\implies$ i): Suppose that $G \cdot v$ is not closed. Pick a closed orbit $G \cdot u \subset \overline{G \cdot v} - G \cdot v$. By a fundamental theorem in GIT, there exists a 1-PS $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v \in G \cdot u$

(this result is used to prove the Hilbert–Mumford criterion). By conjugating λ, we may assume that $\lambda(S^1) \subset K$. The $\lambda(\mathbb{G}_m)$-action on $V = \mathbb{C}^{n+1}$ is completely reducible; that is, we have a weight space decomposition

$$V = \bigoplus_{r \in \mathbb{Z}} V_r \quad \text{where} \quad V_r := \{w \in V : \lambda(t) \cdot w = t^r w\}$$

and this decomposition is orthogonal with respect to H. We can write $v = \sum_r v_r$ in terms of this decomposition. Since $\lim_{t \to 0} \lambda(t) \cdot v \in G \cdot u \subset \overline{G \cdot v} - G \cdot v$, it follows that

(1) $v_r = 0$ for all $r < 0$ (for the limit to exist),
(2) $v_r \neq 0$ for some $r \geq 0$ (for the limit not to lie in $G \cdot v$).

Let us consider the infinitesimal action of

$$A := \frac{d}{dt} \lambda(\exp 2it)|_{t=0} \in \mathfrak{k}.$$

The infinitesimal action of A on v_r is then $A_{v_r} = Av_r = 2irv_r$. Hence,

$$H(Av, v) = \sum_{r,s} H(Av_r, v_s) = \sum_r 2irH(v_r, v_r),$$

where in the second equality, we use the fact that the decomposition is H-orthogonal. Then

$$\mu([v]) \cdot A = \frac{H(Av, v)}{2i||v||^2} = \frac{1}{||v||^2} \sum_{r \geq 0} rH(v_r, v_r) > 0,$$

where the sum is taken over $r \geq 0$ due to (1) and it is positive due to (2). It follows from Lemma 3.4 that e is not a critical point of p_v. We can repeat the above argument replacing v by $g \cdot v$ to prove that e is also not a critical point of $p_{g \cdot v}$. Hence, we conclude that p_v does not have a critical point. □

Remark 3.6. In fact, one can show that the critical values of p_v are non-negative, i.e. p_v is convex and so every critical point is a minimum.

Remark 3.7. As G is the complexification of K, every element $g \in G$ has a Cartan decomposition $g = k \exp(iA)$ for $k \in K$ and $A \in \mathfrak{K}$ (for example every invertible matrix $M \in \mathrm{GL}_n(\mathbb{C})$ has a decomposition $A = UH$ as a product of a unitary matrix U and a Hermitian matrix H).

Proof. (The first version of the Kempf-Ness Theorem.)

Let $v \in \mathbb{C}^{n+1} - \{0\}$ lie over $x = [v] \in \mathbb{P}^n$. Since we will deduce *i)* from *ii)*, we start with the proof of *ii)*.

ii) By Lemma 3.4, we have $g \cdot [v] \in \mu^{-1}(0)$ if and only if p_v has a critical point at g. By Lemma 3.5, this holds if and only if $G \cdot v \subset \mathbb{C}^{n+1}$ is closed. Then by the topological criterion for polystability, $G \cdot v$ is closed if and only if $[v] \in \mathbb{P}^n$ is polystable.

i) $[v] \in \mathbb{P}^n$ is semistable if and only if there is a polystable orbit $G \cdot x'$ in $\overline{G \cdot x}$. By *ii)*, this is the case if and only if $\overline{G \cdot x} \cap \mu^{-1}(0) \neq \emptyset$.

iii) We need to show that if x and y are both points in the same polystable G-orbit with $\mu(x) = \mu(y) = 0$, then x and y belong to the same K-orbit. We can write $x = gy$ and let $v, u \in \mathbb{C}^{n+1}$ be points such that $[v] = x$ and $[u] = y$; then p_u and p_v both attain their minimum at e and $p_u(e) = p_v(e) = p_u(g)$. Consider the Cartan decomposition $g = k \exp(iA)$. As the Hermitian inner product H and its associated norm $|| - ||^2$ are K-invariant, we have

$$p_u(g) = ||g \cdot u||^2 = || \exp(iA) \cdot u||^2 = p_u(\exp(iA)).$$

Since p_u is convex, $p_u(\exp(itA)) \leq p_u(\exp(iA)) = p_u(e)$ for $t \in [0, 1]$. But $p_u(e)$ is the unique global minimum of the convex function p_u and so this must be an equality; that is, p_u is not strictly convex along $\exp(itA)$ or, equivalently,

$$0 = \frac{d^2}{dt^2} p_u(\exp(itA))|_{t=0} = \frac{\pi}{||u||^2} H(Au, Au).$$

Hence, the infinitesimal action $A_y = Au$ is zero and thus $iA \in i\mathfrak{K}_y \subset \mathfrak{g}_y = \mathrm{Lie}\, G_y$. Therefore

$$x = g \cdot y = k \exp(iA) \cdot y = k \cdot y,$$

which proves that x and y belong to the same K-orbit. □

3.4. **The proof of the main version and corollaries.** Now we prove the main version of the Kempf-Ness theorem as stated in the introduction as a Corollary of the first version. A nice exposition of the proof is also found in [7] §8.

Theorem 3.8 (The Kempf–Ness theorem [2]). *The inclusion $\mu^{-1}(0) \subset X^{ss}$ induces a homeomorphism*

$$\mu^{-1}(0)/K \to X//G.$$

Proof. First, we note that the inclusion $\mu^{-1}(0) \subset X^{ss}$ given by the first version of the Kempf–Ness theorem induces a continuous map

$$\begin{array}{ccc} \mu^{-1}(0) & \hookrightarrow & X^{ss} \\ \downarrow & & \downarrow \\ \mu^{-1}(0)/K & \longrightarrow & X//G, \end{array}$$

by the universal property of the topological quotient.

We claim that as sets these two quotients are isomorphic. The set of k-points of the GIT quotient $X//G$ is isomorphic to the set of polystable orbits $X^{ps}(k)/G$. By parts ii) and iii) of the first version of the Kempf-Ness theorem, every polystable orbit meets the zero level set $\mu^{-1}(0)$ in a unique K-orbit and so this gives the required bijection of sets.

As $\mu^{-1}(0)$ is a closed subset of a compact space, it is compact and so is the symplectic reduction $\mu^{-1}(0)/K$. Hence, the map $\mu^{-1}(0)/K \to X//G$ is a continuous bijection from a compact space to a Hausdorff space and so is a homeomorphism. □

Remark 3.9. The inverse of the map $\mu^{-1}(0)/K \to X//G$ can be constructed using the negative gradient flow of the norm square of the moment map; let us explain the idea. If we fix a K-invariant norm on $\mathfrak{k}^*$, then we can consider the negative gradient flow of the norm square of the moment map $||\mu||^2 : X \to \mathbb{R}$. Every point flows to a point in the critical set of $||\mu||^2$ and in this way we obtain a Morse-type stratification of X indexed by components of the critical set. The critical set for the lowest Morse stratum is the zero level set $\mu^{-1}(0)$.

On the algebraic side, one use a normalised Hilbert–Mumford weight to stratify the unstable locus $X - X^{ss}$ following work of Kempf [3] and Hesselink [1]. Let us recall that the Hilbert–Mumford criterion: a closed point $x \in X$ is semistable if and only if $\mu(x, \lambda) \geq 0$ for all 1-parameter subgroups $\lambda : \mathbb{G}_m \to G$, where $\mu(x, \lambda)$ is the Hilbert–Mumford weight. The norm on $\mathfrak{k}^*$ is used to construct a normalised Hilbert–Mumford weight. Then for each unstable orbit one associates a conjugacy class of (rational) 1-parameter subgroups which minimise the normalised Hilbert–Mumford weight and so are 'most responsible' for the instability of these points; this is Kempf's notion of an adapted 1-PS [3]. The lowest stratum in the GIT stratification is the semistable set X^{ss}.

A result of Kirwan [4] and Ness [8] says that these two stratifications agree. In particular, the lowest Morse stratum for $||\mu||^2$ coincides with X^{ss} and so the negative gradient flow induces a retraction $X^{ss} \to \mu^{-1}(0)$ which determines an inverse to the map $\mu^{-1}(0)/K \to X//G$.

We give a proof of a second very useful corollary.

Corollary 3.10. *The origin is a regular value of μ if and only if $X^{ss} = X^s$.*

Proof. The origin is a regular value of μ if and only if for each $x \in \mu^{-1}(0)$ the stabiliser group K_x is finite. Since $G = K_{\mathbb{C}}$, the stabiliser K_x is finite if and only if the stabiliser $G \cdot x$ is finite. Then by the first version of the Kempf–Ness theorem, it follows that G_x is finite for all $x \in \mu^{-1}(0)$ if and only if every polystable orbit has full dimension, that is, if and only if $X^s = X^{ps}$. As every semistable orbit $G \cdot x$ which is not closed contains a unique polystable orbit in its closure of dimension strictly less than $G \cdot x$, it follows that $X^s = X^{ps}$ if and only if $X^s = X^{ss}$. □

Remark 3.11. If K acts freely on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/K$ is smooth. In this case, 0 is a regular value of μ and thus $X^{ss} = X^s$. Since the G-action on X^{ss} is free, it follows from Luna's étale slice theorem that $X//G \cong X^{ss}/G$ is smooth.

References

1. W. H. Hesselink, *Uniform instability in reductive groups*, J. Reine Angew. Math. **304** (1978), 74–96.
2. G. Kempf and L. Ness, *The length of vectors in representation spaces*, Algebraic Geometry, Lecture Notes in Mathematics, vol. 732, Springer Berlin / Heidelberg, 1979, pp. 233–243.
3. G. R. Kempf, *Instability in invariant theory*, Ann. of Math. **108** (1978), no. 2, 299–316 (English).
4. F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, Mathematical Notes, no. 31, Princeton University Press, 1984.
5. J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. **5** (1974), 121–130.
6. K. Meyer, *Symmetries and integrals in mechanics*, Dynamical systems (M. Peixoto, ed.), Academic Press New York, 1973, pp. 259–273.
7. D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Springer, 1993.
8. L. Ness, *A stratification of the null cone via the moment map (with an appendix by D. Mumford)*, Amer. J. Math. **106** (1984), no. 6, 1281–1329.
9. A. L. Onishchik and E. B. Vinberg, *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras*, Encyclopedia of mathematical sciences, vol. 41, Springer-Verlag, 1990.
10. C. Woodward, *Moment maps and geometric invariant theory*, Les cours du CIRM **1** (2010), no. 1, 55–98.