Complex Analysis SS15
Fachbereich Mathematik und Informatik
J-Prof. Victoria Hoskins
Teaching Assistant: Alejandra Rincón

Complex analysis
Summer semester 2015

Exercise Sheet 2

Hand-in date: 12:00, Friday 24th April.

Exercise 1.

a) At which points are the modulus function $| - | : \mathbb{C} \to \mathbb{R}$ and the principal branch of the argument function $\mathrm{Arg} : \mathbb{C}^* \to (-\pi, \pi]$ continuous?

b) Show that the principal branch of the logarithm $\mathrm{Log} : \mathbb{C}^* \to \mathbb{C}$ is discontinuous on $\mathbb{R}_{<0} := \{z = x + iy : y = 0, x < 0\}$ and continuous on $\mathbb{C}_- := \mathbb{C}^* - \mathbb{R}_{<0}$.

c) For $z, w \in \mathbb{C}^*$, determine the value of $\mathrm{Log}(z \cdot w) - \mathrm{Log}(z) - \mathrm{Log}(w)$.

Exercise 2.

a) For a function $f : \mathbb{C} \to \mathbb{C}$, show the following are equivalent formulations of continuity of f at a point $a \in \mathbb{C}$.
   (i) For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - f(a)| < \epsilon$ for $|z - a| < \delta$.
   (ii) For all convergent sequences $z_n \to a$, the sequence $f(z_n)$ converges to $f(a)$.

b) Show that $f : \mathbb{C} \to \mathbb{C}$ is continuous if and only if for every open set $U \subset \mathbb{C}$, the preimage $f^{-1}(U) := \{z \in \mathbb{C} : f(z) \in U\}$ is open.

c) A subset $D \subset \mathbb{C}$ is said to be compact if whenever $D \subset \bigcup_{i \in I} U_i$, for a family of open sets U_i, then there exists a finite subset $I' \subset I$ such that $D \subset \bigcup_{i \in I'} U_i$. Let $D \subset \mathbb{C}$ be compact and $f : D \to \mathbb{C}$ be continuous; then show $f(D)$ is compact.

d) Give an alternative proof that the argument function $\mathrm{Arg} : S^1 \to (-\pi, \pi]$ is not continuous.

Exercise 3. (The quotient rule.)
Let $D \subset \mathbb{C}$ and $f, g : D \to \mathbb{C}$ be functions on this subset with $g(z) \neq 0$ for all $z \in D$. If f and g are holomorphic at $a \in D$ and $g'(a) \neq 0$, show that f/g is also holomorphic at a and

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}.$$

Exercise 4. Describe the sets on which the following functions are holomorphic and find the derivatives at all holomorphic points.

a) $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \overline{z}$,

b) $g : \mathbb{C} \to \mathbb{C}$ given by $g(z) = z|z|^2$,

c) $h : \mathbb{C} \to \mathbb{C}$ given by $h(x + iy) = x^3y^2 + ix^2y^3$.