Complex Analysis SS15
Fachbereich Mathematik und Informatik
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# Complex analysis
# Summer semester 2015

## Exercise Sheet 11

**Hand-in date:** 12:00, Friday 26th June.

**Exercise 1.** In this exercise, we will prove the Fundamental Theorem of Algebra using homotopies of paths in $\mathbb{C}^*$. Let $P$ be a polynomial of degree $n > 0$ and suppose for a contradiction that $P$ has no complex root. For $z \in \mathbb{C}$, consider the path $\gamma_z : [0,1] \to \mathbb{C}^*$ given by

$$\gamma_z(t) = \frac{P(z\exp(2\pi i t))/P(z)}{|P(z\exp(2\pi i t))/P(z)|}.$$

a) For $z \in \mathbb{C}$, prove that $\gamma_z$ is homotopic in $\mathbb{C}^*$ to the constant path $\gamma_0$ by varying $z$.
b) Show there exists $r > 0$ such that for $|z| > r$, $\gamma_z$ is homotopic in $\mathbb{C}^*$ to the $n$-circle path $c_n : [0,1] \to \mathbb{C}^*$, $c_n(t) = \exp(2\pi i n t)$ by considering the family of polynomials $P_l(z) := z^n + l(P(z) - z^n)$ for $l \in [0,1]$.
c) Use these facts to give a proof of the Fundamental Theorem of Algebra.

**Exercise 2.** Let $S$ be a discrete subset of $\mathbb{C}$ and let $f : \mathbb{C} - S \to \mathbb{C}$ be an analytic function such that all points in $S$ are isolated singularities of $f$.

a) If $f$ is even, show that $\mathrm{Res}\,(f; -a) = -\mathrm{Res}\,(f, a)$.
b) If $f$ is odd, show that $\mathrm{Res}\,(f; -a) = \mathrm{Res}\,(f, a)$.
c) If $f$ takes real values on $\mathbb{R}$, show that $\mathrm{Res}\,(f; \overline{a}) = \overline{\mathrm{Res}\,(f, a)}$.
d) If there exists $w$ such that $f(z+w) = f(z)$ for all $z$, show that $\mathrm{Res}(f; a+w) = \mathrm{Res}(f, a)$.

**Exercise 3.** Use the Residue Theorem to evaluate the following integrals.

a)
$$\int_{\partial D_2(0)} \frac{1}{(z-4)(z^3-1)}\,dz$$

b)
$$\int_{\partial D_1(0)} \cot(z)\,dz$$

c)
$$\int_{\partial D_2(0)} z\exp(3/z)\,dz$$

d)
$$\int_{\partial D_1(0)} \sin(1/z)\,dz$$

**Exercise 4.** Let $D$ be a simply connected domain and let $f$ be a function on $D$ that is analytic except at a set of isolated singularities $S$. Prove that $f$ has a primitive on $D - S$ if and only if all residues of $f$ vanish.