SEMINAR ON $\mathbb{A}^1$-HOMOTOPY THEORY OF SCHEMES

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Our seminar takes place on Wednesday afternoons, 16:15 - 17:45 in Arnimallee 3, SR 005. The first seminar will take place on the 16th April 2014. All are welcome to attend and please let me know if you would like to give a talk.

Introduction

In algebraic topology, the notion of homotopy equivalence is important as many invariants (for example, the singular homology groups) are homotopy invariant. In algebraic geometry, the idea of $\mathbb{A}^1$-homotopy theory is to do homotopy theory for schemes where the affine line plays the role of the unit interval. The framework for doing this is to work with the '$\mathbb{A}^1$-homotopy category' by abstracting ideas from algebraic topology. The basic idea is that we want to construct this category by inverting certain morphisms, but the details are more subtle: we need to enlarge the category of smooth varieties so we can take colimits of diagrams and, moreover, introduce a model category structure that allows us to invert the desired class of morphisms. Several talks are devoted to the careful construction of the (unstable) $\mathbb{A}^1$-homotopy category.

In this seminar, we focus our attention on the unstable theory as this suffices for us to achieve our main aim: to provide a homotopical classification of vector bundles on a smooth k-scheme. We recall that in algebraic topology, the classifying space of $\mathrm{GL}(n, \mathbb{C})$ can be constructed as the infinite grassmannian Gr_n of n-planes and, moreover, isomorphism classes of (topological) rank n complex vector bundles on a space X are in bijection with homotopy classes of maps from X to Gr_n. Our goal is to provide an algebraic version of this statement using the (unstable) $\mathbb{A}^1$-homotopy category. This aim seems appropriate for our seminar as it is closely linked to the research interests of our group and will also give us a good introduction to the ideas and potential of $\mathbb{A}^1$-homotopy theory.

The plan for the seminar is to start from the basics and keep in mind the following goals.

(1) To motivate the desire for an $\mathbb{A}^1$-homotopy theory of k-schemes from the point of view of homotopy theory in algebraic topology.
(2) To construct the $\mathbb{A}^1$-homotopy category (for this, we will need to make a digression into Quillen's theory of homotopical algebra) and understand why this category is the right one to consider.
(3) To provide a homotopical classification of vector bundles on a smooth k-scheme.

Description of the talks

Talk 1: Introduction. We recall the notion of homotopy in algebraic topology and state the classification of principal G-bundles in terms of homotopy classes of maps to the classifying space BG (this will be proved in Talk 2). Our focus is on vector bundles and we recall that the classifying space $B\mathrm{GL}(n, \mathbb{C})$ can be described as the infinite Grassmannian of complex n-planes Gr_n. We also discuss the stable version of this result that relates the topological Grothendieck group of (complex) vector bundles on X to homotopy classes of maps from X to $\mathrm{Gr} \times \mathbb{Z}$ where Gr is the colimit of Gr_n over n.

We then give an overview of the ideas involved in constructing a homotopy theory in algebraic geometry. The first point to address is what should play the role of the unit interval $[0, 1]$. The algebraic analogue of this is the affine line $\mathbb{A}^1$ and we describe some of the similarities and differences between these two different interval objects. Using the affine line as our interval object, we give the naive notion of homotopy for schemes (known as strict $\mathbb{A}^1$-homotopy equivalences). A more subtle question, is which category of 'spaces' should we work with and is this category sufficiently well-behaved. The category of smooth k-varieties $\mathcal{V}$ is too small (for example, not all diagrams have colimits) and so we consider the larger category of sheaves on $\mathcal{V}$ (for a suitable topology). The choice of topology is important: we take the Nisnevich topology which is finer than the Zariski topology and weaker than the étale topology as it has many useful properties of both. We want to invert a certain class of morphisms and one way to do this is to use techniques from homotopical algebra (e.g. model structures, see Talk 3). In order to be able to introduce a model structure and do homotopy theory, we must consider simplicial sheaves on $\mathcal{V}$ in the Nisnevich topology. The (unstable) $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ is constructed by inverting a class of morphisms that essentially play the role of homotopies (the so-called $\mathbb{A}^1$-weak equivalences). Morphism groups in the $\mathbb{A}^1$-homotopy category are notoriously difficult to calculate. For example, even calculating the maps from $\operatorname{Spec} k$ to a variety X is problematic; conjecturally this should agree with the set of k-points $X(k)$ modulo naive $\mathbb{A}^1$-homotopy.

We end by stating the goal of the seminar: the homotopical classification of algebraic vector bundles. We emphasise the similarities with the classification in algebraic topology.

Talk 2 : Classification of principal bundles in algebraic topology. The goal of this talk is to prove, for a topological group G, that there is a classifying space BG such that homotopy classes of maps from a given (paracompact) space X to BG are in bijective correspondence with the set of isomorphism classes of principal G-bundles on X

$$[X, BG] \cong \operatorname{Bun}_G(X).$$

The first step is to prove the homotopy invariance of fibre bundles i.e. given a fibre bundle $E \to B$ and two homotopic maps $f_i : X \to B$, we have as isomorphism $f_0^*(E) \cong f_i^*(E)$. For a fixed principal G-bundle $P \to B$, pulling back P along a map $X \to B$ gives a well defined map

$$\Phi_P : [X, B] \to \operatorname{Bun}_G(X).$$

A principal G-bundle $p : EG \to BG$ is universal if the induced map $\Phi_{EG} : [X, BG] \to \operatorname{Bun}_G(X)$ is a bijection for all X. Hence, the goal of this talk is to prove the existence of a universal principal G-bundle (and, time permitting, that the classifying space BG is unique up to homotopy type). A key ingredient is the recognition principle: a principal G-bundle $p : P \to B$ whose total space P is aspherical (i.e. its homotopy groups are all trivial) is universal.

For $G = \mathrm{GL}(n, \mathbb{C})$ and $G = \mathrm{U}(n)$, we realise BG as the infinite Grassmannian of complex n-planes Gr_n (the observation that BG coincides for both groups reflects the fact that every complex vector bundle admits a Hermitian metric). Then we deduce a construction of BG for all subgroups $G \subset \mathrm{GL}(n, \mathbb{C})$. Finally, we describe stabilisation of vector bundles in algebraic topology and the representation of the topological Grothendieck group K_0^{top} in homotopy theory.

If time permits, we'll give a few corollaries: i) if $f : X \to Y$ is a homotopy equivalence, it induces isomorphisms $\operatorname{Bun}_G(X) \cong \operatorname{Bun}_G(Y)$, ii) a fibre bundle over a contractible space is trivial, and iii) complex line bundles are classified via $\operatorname{Bun}_{\mathrm{U}(1)}(X) \cong [X, \mathbb{CP}^\infty] \cong H^2(X, \mathbb{Z})$ and real line bundles are classified via $\operatorname{Bun}_{\mathrm{O}(1)}(X) \cong [X, \mathbb{RP}^\infty] \cong H^2(X, \mathbb{Z}_2)$.

References. The literature on this topic is vast; for example, see [2], Chapter 1 for the homotopical classification and [4], Chapter 9 for stabilisation results for vector bundles.

Talk 3: An introduction to homotopical algebra. We define the simplicial category Δ and, for any category $\mathcal{C}$, the category of simplicial (resp. cosimplicial) objects in $\mathcal{C}$ denoted $\Delta^{\mathrm{op}}\mathcal{C}$ (resp. $\Delta\mathcal{C}$). For the category of topological spaces $\mathcal{T}op$, we construct a cosimplicial topological space $\Delta_{\mathcal{T}op}$ and use this to define for any topological space X a simplicial set $S(X) \in \Delta^{\mathrm{op}}\mathcal{S}et$ called the singular simplicial set of X. This functor $S : \mathcal{T}op \to \Delta^{\mathrm{op}}\mathcal{S}et$ has a left adjoint, the realisation functor, which we also describe.

We give the definition of a model structure on a category $\mathcal{C}$ and its associated homotopy category $\operatorname{Ho}\mathcal{C}$ obtained by inverting the weak equivalences. We also define fibrant and cofibrant objects in a model category and state an extremely useful result that allows us to compute morphism groups in the homotopy category between cofibrant and fibrant objects. Examples: the model structure on the category of topological spaces $\mathcal{T}op$; the model structure on the category of simplicial sets $s\mathcal{S}et := \Delta^{\mathrm{op}}\mathcal{S}et$. For example, every simplicial set is cofibrant and simplicial groups are fibrant. We see that there is an equivalence of categories between $\mathcal{H} := \mathrm{Ho}\,\mathcal{T}op$ and $\mathcal{H}_s := \mathrm{Ho}\, s\mathcal{S}et$.

Finally, for a model category $\mathcal{C}$, we give a notion for a cosimplicial object $\Delta^\bullet$ of $\mathcal{C}$ to be compatible with the model structure on $\mathcal{C}$. We show such an object defines a 'simplicial homotopy relation' on morphism groups $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and let $\pi(X,Y)$ denote the quotient by this relation. We end with an important result: for X cofibrant and Y fibrant, we have $\pi(X,Y) = \mathrm{Hom}_{\mathrm{Ho}\,\mathcal{C}}(X,Y)$.
References The basic notions are given in [8] p369–370 and §2.3 (p373–378); for a more detailed approach, see [3] §1.1-1.2, §2.4 and §3.1-3.2 and see also [1] §2. A concise description of the model category structures on simplicial sets and topological spaces is given in [6], §17.

Talk 4: Sheaves in the Nisnevich topology. Let $\mathcal{V}$ denote the category of smooth, separated, finite type schemes over a perfect field k. We define the étale, Nisnevich and Zariski topologies on $\mathcal{V}$. For a Grothendieck topology τ, we recall the definition of the category of sheaves of sets in the τ-topology, denoted $\mathcal{S}h_\tau(\mathcal{V})$. We note that we have a sequence of full embeddings

$$\mathcal{V} \subset \mathcal{S}h_{\text{ét}}(\mathcal{V}) \subset \mathcal{S}h_{\mathrm{Nis}}(\mathcal{V}) \subset \mathcal{S}h_{\mathrm{Zar}}(\mathcal{V}) \subset \mathcal{P}resh(\mathcal{V})$$

where the leftmost inclusion is given by sending a scheme to its functor of points. We give a comparison between the Zariski, Nisnevich and étale topologies and see that the Nisnevich topology has many useful properties (it sits between the Zariski and étale topologies and has many desirable features of both topologies).

We then define distinguished squares for the Nisnevich topology. One of the many reasons for working with the Nisnevich topology is that one can check if a presheaf is a sheaf using distinguished squares (cf. [8] Lemma 2.1.6). We define τ-points and τ-neighbourhoods of τ-points in $\mathcal{V}$; this allows us to define the fibre of a presheaf. We also show distinguished squares that define a square in $\mathcal{S}h_{\mathrm{Nis}}(\mathcal{V})$ that is both cartesian and cocartesian give rise to a canonical isomorphism of sheaves (cf. [8] Lemma 2.1.13).
References For an overview, see [8] §2.1 (p364–368) and also p371–372. See also the start of Section 3 in [10] (p94-95) which gives several properties of the Nisnevich topology that explain its use in the construction of the $\mathbb{A}^1$-homotopy category.

Talk 5: The simplicial homotopy category of sheaves. We define the category $\Delta^{\mathrm{op}}\mathcal{S}h_\tau(\mathcal{V})$ of simplicial sheaves in the τ-topology (we are ultimately interested in the case when $\tau = \mathrm{Nis}$). We see that we can associate to any sheaf, a corresponding constant simplicial sheaf. For any set S, we can consider the associated constant sheaf S and similarly for simplicial sets we obtain a simplicial sheaf; i.e. $s\mathcal{S}et \hookrightarrow \Delta^{\mathrm{op}}\mathcal{S}h_\tau(\mathcal{V})$. The n-simplex Δ^n and n-sphere $S^n = \Delta^n/\partial\Delta^n$ can then be defined in the category of simplicial sheaves $\Delta^{\mathrm{op}}\mathcal{S}h_\tau(\mathcal{V})$. We let $\emptyset$ and $*$ denote the initial and final objects in $\Delta^{\mathrm{op}}\mathcal{S}h_\tau(\mathcal{V})$. We remark that one can analogously define a category of pointed simplicial sheaves in order to define smash products. Then we introduce a model structure on $\Delta^{\mathrm{op}}\mathcal{S}h_\tau(\mathcal{V})$ such that the weak equivalences are those whose fibres at τ-points are weak equivalences of simplicial sets. We define the simplicial homotopy category $\mathcal{H}_s^\tau(\mathcal{V})$ of sheaves in the τ-topology to be the associated homotopy category. We also note that there is a pointed analogue, denoted $\mathcal{H}_{s,\bullet}^\tau(\mathcal{V})$.

We then make a digression back into the theory of homotopical algebra and give notions of Quillen functors and derived functors (we remark that these can be computed by taking (co)fibrant resolutions). We show the following functors are Quillen functors: i) the functor given by taking a product with a fixed simplicial sheaf and ii) the functor given by taking the smash product with a fixed pointed simplicial sheaf.

Finally, we describe the B.G. property for a simplicial presheaf (a notion that was first introduced in the Zariski topology by Brown and Gersten). We remark that fibrant simplicial sheaves in the Nisnevich topology have the B.G. property (this generalises a result from last week: the sheaf property in the Nisnevich topology can be checked on distinguished squares). We give a result that, for a pointed simplicial sheaf $\mathcal{X}$ with the B.G. property, allows us to relate certain morphism groups in $\mathcal{H}_{s,\bullet}^{\mathrm{Nis}}(\mathcal{V})$ with the homotopy groups of the simplicial set $\mathcal{X}(U)$ for

$U \in \mathcal{V}$ (this follows from a general result in homotopical algebra that describes the morphisms between cofibrant and fibrant objects).

References. For an overview, see [8] §2.3 (p378–382). Further details can be found in [10].

Talk 6: The $\mathbb{A}^1$-homotopy category of smooth k-schemes $\mathcal{H}(k)$. Henceforth, we work with the Nisnevich topology and so (simplicial) sheaf means (simplicial) sheaf for the Nisnevich topology. We start by recalling the naive homotopy equivalence relation, denoted $\sim_{\mathbb{A}^1}$ given by strict $\mathbb{A}^1$-homotopy equivalences; then we define a more general notion of $\mathbb{A}^1$-weak equivalences. An object $\mathcal{X}$ in $\Delta^{\mathrm{op}}\mathcal{S}h_{\mathrm{Nis}}(\mathcal{V})$ is $\mathbb{A}^1$-local if, under the natural projection $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$, the morphisms groups from $\mathcal{X} \times \mathbb{A}^1$ and $\mathcal{X}$ agree in the simplicial homotopy category $\mathcal{H}_s^\tau(\mathcal{V})$. Using this notion, we define $\mathbb{A}^1$-weak equivalences; these are the morphisms we invert to construct the $\mathbb{A}^1$-homotopy category and these include simplicial weak equivalences and projection maps $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$. Morel and Voevodsky prove that there is a model structure on $\Delta^{\mathrm{op}}\mathcal{S}h_{\mathrm{Nis}}(\mathcal{V})$ whose weak equivalences are $\mathbb{A}^1$-weak equivalences and then construct the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ of smooth k-schemes as the homotopy category of this model category. Again, we remark that there is a pointed version $\mathcal{H}_\bullet(k)$ of $\mathcal{H}(k)$ which has a suspension functor (given by taking a smash product with S^1).

We describe two natural cosimplicial objects that are compatible with this model structure: the standard cosimplicial simplex and an algebraic version $\Delta^\bullet_{\mathrm{alg}}$. For the algebraic version, we observe that the associated simplicial homotopy equivalence relation on morphism groups coincides with the the notion of strict $\mathbb{A}^1$-homotopy equivalence $\sim_{\mathbb{A}^1}$. We recall from the talk on model categories that for $\mathcal{X}$ cofibrant and $\mathcal{Y}$ fibrant

$$\mathrm{Hom}_{\mathcal{H}(k)}(X, Y) = \pi(X, Y) := \mathrm{Hom}_{\Delta^{\mathrm{op}}\mathcal{S}h_{\mathrm{Nis}}(\mathcal{V})}(X, Y)/\sim_{\mathbb{A}^1}.$$

Let $\mathcal{H}_{s,\mathbb{A}^1}^{\mathrm{Nis}}(\mathcal{V}) \subset \mathcal{H}_s^{\mathrm{Nis}}(\mathcal{V})$ be the full subcategory consisting of $\mathbb{A}^1$-local simplicial sheaves; then, this inclusion has a left adjoint, the $\mathbb{A}^1$-localisation functor $L_{\mathbb{A}^1} : \mathcal{H}_s^{\mathrm{Nis}}(\mathcal{V}) \to \mathcal{H}_{s,\mathbb{A}^1}^{\mathrm{Nis}}(\mathcal{V})$ which induces an equivalence of categories $\mathcal{H}(k) \to \mathcal{H}_{s,\mathbb{A}^1}^{\mathrm{Nis}}(\mathcal{V})$.

When we have a complex (resp. real) embedding of k, we see there is an induced functor $\mathcal{H}(k) \to \mathcal{H}$ (resp. two non-isomorphic induced functors). Analogously to the notation in algebraic topology, we denote the morphism group in the homotopy category by

$$[\mathcal{X}, \mathcal{Y}] := \mathrm{Hom}_{\mathcal{H}(k)}(\mathcal{X}, \mathcal{Y});$$

in general, it is very hard to compute these groups. Morel and Voevodsky conjecture that for affine X the morphism group from X to Y in $\mathcal{H}(k)$ is the naive $\mathbb{A}^1$-homotopy classes of morphisms from X to Y.

References. For an overview, see [8] §3.1 (p383–386) and for the proofs, see [10]. We skip the proof of the model structure, as this is quite dry and technical.

Talk 7 : K_0 and K_1 for rings. We recall the construction of the Grothendieck group K_0 of an exact category $\mathcal{A}$ (i.e. an additive category that has a class of 'short exact sequences', but is not necessarily abelian). For a ring R, we define $K_0(R)$ to be the Grothendieck group of the category of finitely generated projective R-modules. For a scheme X, we define $K_0(X)$ to be the Grothendieck group of the category of locally free $\mathcal{O}_X$-modules. For a Noetherian scheme X, we define $G_0(X)$ (often also denoted K_0') to be the Grothendieck group of the category of coherent sheaves on X and observe that if X is also regular then $G_0(X) = K_0(X)$. We define the reduced K_0-group $\tilde{K}_0(X)$ to be the kernel of the rank map $K_0(X) \to H^0(X, \mathbb{Z})$ and note that there are maps from the set $\Phi_n(X)$ of rank n vector bundles over X to $\tilde{K}_0(X)$ given by $E \mapsto [E] - [\mathcal{O}_X^n]$.

It is not necessary for us to define the higher K-theory groups of a scheme; however, we will need to define K_1 of a ring. Fortunately, we can bypass the formalism used by Quillen and instead define K_1 of a ring R via the short exact sequence

$$1 \to [\mathrm{GL}(R), \mathrm{GL}(R)] \to \mathrm{GL}(R) \to K_1(R) \to 1$$

where $\mathrm{GL}(R) = \mathrm{colim}_n\, \mathrm{GL}_n(R)$. In particular $K_1(R)$ is an abelian group and every homomorphism from $\mathrm{GL}(R)$ to an abelian group factors through $K_1(R)$. When $R = F$ is a field, we see that $\mathrm{SL}(F) = [\mathrm{GL}(F), \mathrm{GL}(F)]$ and $K_1(F) = F^\times$. We prove Whitehead's lemma which states that the commutator group $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is the group of elementary matrices $E(R) \subset \mathrm{GL}(R)$.

Finally, we define K_0 and K_1-regularity of a ring and prove that if R is a regular ring then it is both K_0 and K_1-regular (in fact K_1-regularity implies K_0-regularity).

References. The main reference is [11], Chapters II and III. For Whitehead's lemma, see [11] III 1.3.3 and for the proof of K_1-regularity of a regular ring, see [11] III 3.8 (and also II 6.5, for K_0-regularity) and [7] p45–48.

Talk 8 : Excision for Picard groups and Karoubi-Villamayor K-theory. We recall that the Picard group of a regular scheme can be calculated from an exact sequence of groups. Then we show that distinguished squares of schemes in $\mathcal{V}$ give rise to an exact sequence relating their Picard groups. We also prove that $K_0(X)$-regularity implies Pic regularity (cf. [7], Lemma 4.1.7).

For a commutative ring R, we construct an associated simplicial group $\underline{\mathrm{GL}}(R)$ and define the Karoubi-Villamayor K-theory groups of R by taking homotopy groups. More precisely, we define $KV_0(R) := K_0(R)$ and $KV_n(R) := \pi_{n-1}\underline{\mathrm{GL}}(R)$, for $n \geq 1$. We see that there is a canonical epimorphism $K_1(R) \to KV_1(R)$ and prove that this is an isomorphism when R is K_1-regular. We deduce some consequences (e.g., [7] Lemma 4.1.12 and Corollary 4.1.15) that show certain cartesian squares of commutative rings give rise to natural long exact sequences in Karoubi-Villamayor K-theory groups KV_n (and, moreover, a long exact sequence mixing the KV_n-groups with the K_0 and K_1-groups).

References. See [7], §4.1.1 and §4.1.4 (see p44–51) and the references therein.

Talk 9: Anodyne extensions I. We introduce an alternative construction of the $\mathbb{A}^1$-homotopy category given by Morel in [7]. The construction of the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ given above follows the construction given by Morel and Voevodsky in [10]; this is the construction used by most working in the field today. Unfortunately, we need to use the alternative construction of Morel in [7] to prove that the infinite Grassmannian is fibrant (cf. Talk 11) which is an important step towards the homotopical classification result. By [7] Remark 2.2.15, both constructions lead to equivalent categories.

As motivation, we give the description of anodyne extensions for simplicial sets. We state Morel's notion of quasi-simplicial categories (a weaker notion than simplicial model categories, but which allow us to invert a class of morphisms to obtain a reasonably behaved homotopy category). This is the tool Morel uses for his construction of the $\mathbb{A}^1$-homotopy category. Morel inverts a class of morphisms in the category of 'k-spaces' $\mathcal{E}_k := \mathrm{Fun}(\mathcal{A}f\mathcal{S}m_k^{\mathrm{op}}, \mathcal{S}et)$; that is, presheaves on the category $\mathcal{A}f\mathcal{S}m_k$ of smooth affine schemes of finite type over k. The quasi-simplicial category structure on $\mathcal{E}_k$ is specified by giving a cosimplicial object $\Delta^\bullet_{\mathrm{alg}}$ and classes of elementary cofibrations and elementary anodyne extensions. We start defining these class of morphisms; the remaining definitions will be given in the following talk.

References. [7] §2.2.1 (p14–20). The notion of anodyne extensions first arose from the model structure on simplicial sets; for example, see [6], 17.5 and [3] §3.2-3.3.

Talk 10: Anodyne extensions II. We continue with the definition of the quasi-simplicial structure on the category of k-spaces $\mathcal{E}_k := \mathrm{Fun}((\mathcal{A}f\mathcal{S}m_k)^{\mathrm{op}}, \mathcal{S}et)$. It remains to define the class of fundamental geometric anodyne extensions (we recall that the class of anodyne extensions is defined so it contains two special classes of morphisms: the fundamental simplicial anodyne extensions and the fundamental geometric anodyne extensions). We define the fundamental geometric anodyne extensions and show that any section of a vector bundle over a k-space is a fundamental geometric anodyne extension.

We define the cofibrations, trivial fibrations and anodyne extensions using theses classes of elementary cofibrations and elementary anodyne extensions. We obtain a description of the cofibrant objects and, in particular, note that every smooth affine scheme and the infinite general linear group are both cofibrant. In Talk 11, we see that the (doubly) infinite Grassmannian is fibrant and in Talk 12, we see that the constant presheaf associated to any set is also fibrant.

Morel constructs $\mathcal{H}(k)$ by inverting all anodyne extensions between cofibrant objects and trivial fibrations in $\mathcal{E}_k$ (the class of inverted maps are referred to as weak equivalences). We show that any naive $\mathbb{A}^1$-homotopy equivalence is a weak equivalence and note (again) that the morphism group in $\mathcal{H}(k)$ from a cofibrant k-space X to fibrant k-space Y are precisely the naive homotopy classes of maps $X \to Y$ (cf. [7] Proposition 2.2.14). Then we compare this construction of Morel with the construction of Morel and Voevodsky covered in Talks 4-6.

Finally, we state a technical result that we will need in the following talks: for an elementary anodyne extension $X \to Y$, the associated simplicial map $\underline{\mathrm{GL}}(A(Y)) \to \underline{\mathrm{GL}}(A(X))$ is a weak equivalence. The idea is to prove $\underline{\mathrm{GL}}(A(Y)) \to \underline{\mathrm{GL}}(A(X))$ is a trivial fibration (and so a weak equivalence) by firstly showing that is true for a fundamental geometric anodyne extensions and then to use an inductive argument to prove the general case. Moreover, it follows that for an elementary anodyne extension as above, we have isomorphisms

$$K_0(A(Y)) \cong K_0(A(X)) \quad \text{and} \quad A(Y)^\times \cong A(X)^\times \quad \text{and} \quad \mathrm{Pic}(X) \cong \mathrm{Pic}(Y).$$

References. [7] §2.2.1 (p14–20) and §4.2.1 (p51–54).

Talk 11 : Homotopic properties of the canonical GL_n-torsor on the Grassmannian $\mathbf{Gr}_n$. We start this talk with a brief digression about vector bundles and GL_n-torsors. For us, an important example is the GL_n-torsor $V_{n,r} \to \mathrm{Gr}_{n,r}$ over the Grassmannian of n-planes in $\mathbb{A}^{n+r}$. By taking the colimit (over r) we get the canonical GL_n-torsor $V_n \to \mathrm{Gr}_n$ over the infinite Grassmannian of n-planes. For $n = 1$, we get the canonical $\mathbb{G}_m$-torsor $\mathbb{A}^* := V_1 \to \mathbb{P}^\infty := \mathrm{Gr}_1$. By taking the colimit of the canonical GL_n-torsors $V_n \to \mathrm{Gr}_n$ over n we get a GL-torsor $V \to \mathrm{Gr}$ over the infinite Grassmannian.

Then the aim of this talk is to prove the following.

(1) the canonical GL-torsor $V \to \mathrm{Gr}$ is a fibration.
(2) the canonical $\mathbb{G}_m$-torsor $\mathbb{A}^* \to \mathbb{P}^\infty$ is a fibration.
(3) the infinite projective space $\mathbb{P}^\infty$ and Grassmannian Gr are both fibrant.

We first show that the infinite general linear group GL and multiplicative group $\mathbb{G}_m$ are fibrant. Then we prove that the GL-torsor $V \to \mathrm{Gr}$ is a fibration (the proof for the canonical $\mathbb{G}_m$-torsor is similar). To prove that Gr is fibrant we need to show, for an elementary anodyne extension $X \to Y$, that and map $X \to \mathrm{Gr}$ can be lifted to Y. The proof of this final part involves the defining properties of Grassmannian as well as the fact (proved last week) that $K_0(A(Y)) \cong K_0(A(X))$. The argument for the infinite projective space is similar.

References. [7] §2.1.5 (p9–13) and §4.2.2 (p54–58).

Talk 12: Homotopic classification of vector bundles on smooth k-schemes. The goal of this talk is to prove that for a smooth k-scheme X, there are natural bijections:

(1) $[X, \mathbb{Z} \times \mathrm{Gr}] \cong K_0(X)$;
(2) $[X, \mathbb{P}^\infty] \cong \mathrm{Pic}(X)$.

We firstly prove the result for a smooth affine k-scheme as these are cofibrant (for Morel's construction). The idea is to use the fact that the morphism group in the homotopy category between a cofibrant and fibrant object is the group of naive homotopy classes of maps (cf. [7], Proposition 2.2.14). Last week we proved that Gr and $\mathbb{P}^\infty$ are fibrant and we show that $\mathbb{Z}$ is also fibrant (see [7], Lemma 4.2.8); therefore the codomains $\mathbb{Z} \times \mathrm{Gr}$ and $\mathbb{P}^\infty$ are both fibrant. Then for affine X we deduce the homotopical classification by computing the naive homotopy groups $\pi(X, \mathbb{Z} \times \mathrm{Gr})$ and $\pi(X, \mathbb{P}^\infty)$ (this computation mostly relies on standard properties of Grassmannians, cf. [7] Proposition 4.2.11).

Unfortunately smooth k-schemes are not always cofibrant (for Morel's construction) and so we need to do some extra work to deduce the homotopic classification in this case. The idea is to use the Jouanolou-Thomason theorem: for any smooth separated k-scheme X, there is a vector bundle $E \to X$ and a E-torsor $T \to X$ such that T is affine (the notion of a torsor under a vector bundle is given in [7] §2.1.5, p13). As $T \to X$ is a torsor under a vector bundle, it follows that $T \to X$ is a weak equivalence (see [7], Corollary 2.3.2) and so $[X, B] \to [T, B]$ is bijective. The idea is then to deduce the case for X from the case for the affine smooth scheme T proved above by using strong $\mathbb{A}^1$-homotopy invariance for K_0; that is, we want to use homotopy invariance to show that $K_0(X) \cong K_0(T)$. We recall that for a regular scheme X we have that $K_0(X \times \mathbb{A}^n) \cong K_0(X)$ (this is strong $\mathbb{A}^1$-homotopy invariance for K_0). It follows that, for a vector bundle $E \to X$, we have $K_0(X) \cong K_0(E)$ by using induction on the number of open sets needed to trivialise with the Mayer-Vietoris property for K_0. More generally, for a torsor $T \to X$ under a vector bundle, we deduce that $K_0(X) \cong K_0(T)$ (and similarly $\mathrm{Pic}(X) \cong \mathrm{Pic}(T)$).

References. See [7] §4.2.3.1-4.2.3.2 (p58–61) and Appendix B.4 for details of the Jouanolou-Thomason theorem.

References

1. P.G. Goerss and J.F. Jardin, *Simplicial homotopy theory*, Progress in mathematics, Birkhauser, 1996.
2. A. Hatcher, *Vector bundles and K-theory*, http://www.math.cornell.edu/ hatcher/VBKT/VBpage.html.
3. M. Hovey, *Model categories*, Mathematical surveys and monographs, vol. 63, AMS, 1999.
4. D. Husemoller, *Fibre bundles*, Graduate texts in mathematics, vol. 20, Springer–Verlag, 1966.
5. M. Levine, *Motivic homotopy theory*, Milan J. Math. **76** (2008), 165–199.
6. J. P. May and K. Ponto, *More concise algebraic topology*, Chicago lectures in mathematics series, University of Chicago Press, 2012.
7. F. Morel, *Homotopy theory of schemes*, SMF/AMS texts and monographs, vol. 12, SMF/AMS, 1999.
8. ———, *An introduction to A^1-homotopy theory*, **15** (2003).
9. ———, *A^1-algebraic topology over a field*, Lecture notes in Mathematics, vol. 2052, Springer, Heidelberg, 2012.
10. F. Morel and V. Voevodsky, *A^1-homotopy theory of schemes*, Publ. Math. I.H.É.S. **90** (2001), 45–143.
11. C. Weibel, *The K-book: an introduction to algebraic K-theory*, Graduate studies in Math., vol. 145, AMS, 2013.

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