Preface

This book currently contains one part, but will be extended over the years. I personally believe this book should and will always remain online and available for free on my personal website. One should consider this book as being “alive”, i.e. frequently edited; you can find the date of the latest edit on the cover page.

Part I is concerned with the theory of complex representations. A first reader should definitely not begin representation theory over an arbitrary field as it is incredibly more subtle and the generalized notions can be more easily understood when the simpler notions applied to complex representations are resting on a solid basis.

The plan for the book is to create a second part concerning representations over an arbitrary field, a third section on locally compact groups and a fourth section on modular representations (which is the theory in characteristic \( p \)). For the moment, this book stays with one part. Any collaboration offer is welcome and can be discussed with me via e-mail.

One can also see that there are no exercises in this book. This is because the author didn't have the time to build a seriously interesting database of exercises together with their solutions. The experienced reader might also notice that I heavily relied on Serre’s classical book “Linear representation of finite groups”, where some of the exercises are solved here as a part of the theory for the sake of completeness. It is the author’s philosophy that parts of the theory a book is built upon should not be given as exercise, to solidify the student’s comprehension of the theory. The exercises should be parallel to the theory, so that the problems required to be solved are problems on which the theory applies, not problems which are required to be solved for the theory to be completely understood. Before building such an exercise database, the book should be somehow a bit more complete so that theory appearing in the future was not given as an exercise in the earlier parts. Therefore, an exercise database is a project for the future.
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Part I

Complex representations of finite groups
Chapter 1

Linear representations

1.1 Generalities

Definition 1.1. Let $V$ be a complex vector space. Denote by $\text{GL}(V)$ the group of isomorphisms of $V$, i.e. $a \in \text{GL}(V)$ means that $a : V \to V$ is a bijective linear map. If $(e_i)_{1 \leq i \leq n}$ is a basis of $V$, there is a bijection

$$a \in \text{GL}(V) \xrightarrow{1:1} (a_{ij}) \in \text{GL}_n(\mathbb{C}).$$

Let $G$ be a finite group with $1 \in G$ the neutral element for the multiplication. A linear representation of $G$ in $V$ is a morphism of groups $\rho : G \to \text{GL}(V)$, i.e. $\forall s, t \in G, \rho(st) = \rho(s)\rho(t)$.

We will often say that $V$ is the representation and write “$V$ is a representation of $G$” instead of naming the triple $(G, V, \rho)$ (this is due to the notion of $\mathbb{C}[G]$-module, c.f. Chapter 3). Under this notation, the corresponding morphism of groups $\rho : G \to \text{GL}(V)$ is often denoted by $\rho^V$. We will write $\rho_s \overset{\text{def}}{=} \rho(s)$. The degree of $(G, V, \rho)$ is the dimension of $V$ as a complex vector space and is denoted by $\deg \rho = \dim V$.

Remark 1.2. For any representation $V$, we have $\rho(1_G) = \text{id}_V$ and $\rho(s^{-1}) = \rho(s)^{-1}$. This is just because $\rho$ is a morphism of groups.

Convention 1.3. When $G$ is a finite group and $s \in G$, we write $1$ to denote the identity element of the group $G$ and we let $\text{ord}(G)$ denote the order or cardinality of $G$. For $s \in G$, we write $\text{ord}(s)$ for the order of $s$, namely

$$\text{ord}(s) \overset{\text{def}}{=} \#\{s^n \mid n \in \mathbb{Z}\}.$$

We restrict ourselves to finite groups represented in finite-dimensional vector spaces.

Definition 1.4. Let $\rho, \rho'$ be two representations of $G$ in $V$ and $V'$, respectively. A $G$-linear map $\tau : V \to V'$ is a linear map such that the following diagram commutes:

$$
\begin{array}{ccc}
V & \xrightarrow{\tau} & V' \\
\downarrow{\rho(s)} & & \downarrow{\rho'(s)} \\
V & \xrightarrow{\tau} & V'
\end{array}
$$

or in other words, $\tau(\rho(s)) = \rho'(\tau(s))$. We also say that $\tau$ is a map of $G$-modules (this expression will be used more later since a map of $G$-modules corresponds to a map of $\mathbb{C}[G]$-modules ; c.f. Chapter 3). Two representations $\rho$ and $\rho'$ of $G$ are isomorphic if there exists a bijective $G$-linear map $\tau : V \to V'$, in which case the inverse map is automatically $G$-linear and we write $\rho \simeq \rho'$. In particular, $\rho$ and $\rho'$ have the same degree.
Example 1.5.

- In matrix form, one can choose a basis of $V$, say $V = \langle e_1, \cdots, e_n \rangle \mathbb{C}$, and write $R_s$ for the matrix form of $\rho_s \overset{def}{=} \rho(s) \in \text{GL}(V)$. Similarly, write $V' = \langle e_1', \cdots, e_n' \rangle \mathbb{C}$ for a basis of $V'$ and $R'_s$ the matrix form of $\rho'_s \overset{def}{=} \rho'(s)$. Saying that $\rho$ and $\rho'$ are isomorphic is equivalent to the existence of a matrix $T \in \text{GL}_n(\mathbb{C})$ such that

$$\forall s \in G, \quad TR_s = R'_sT.$$ 

- A representation of degree 1 of $G$ is the same as a morphism $\rho : G \to \mathbb{C}^\times$ since we have a natural isomorphism $\text{GL}_1(\mathbb{C}) \simeq \mathbb{C}^\times$ (naturality is as functors from fields to groups). Therefore, for any $s \in G$, we have $\rho_s^{\text{ord}(s)} = 1$. In particular, $\rho_s$ is a root of unity and has complex norm 1. If $\rho_s = 1$ for all $s \in G$, we call this 1-dimensional representation the trivial representation of $G$.

- Set $g = \text{ord}(G)$ and let $V$ be a vector space with $\dim V = g$. Take a basis $\{e_t\}_{t \in G}$. For every $s \in G$, define

$$\lambda_s : V \to V$$

$$e_t \mapsto e_{st}.$$ 

This gives us a map

$$\rho : G \to \text{GL}(V)$$

$$s \mapsto \rho_s$$

called the left regular representation of $G$ and is denoted by $\mathbb{C}[G]$. It has degree $g$ by construction. Since $\lambda_s(e_1) = e_s$, the images of $e_1$ form a basis for $V$. Conversely, let $W$ be a representation of $G$, and suppose that there exists a vector $w \in W$ such that $\{\lambda_s(w)\}_{s \in G}$ is a basis for $W$. Then we conclude that $W$ is isomorphic to the regular representation of $G$. (The isomorphism $\tau : V \to W$ is given by $e_s \mapsto \lambda_s(w)$.)

Alternatively, one can define the right regular representation of $G$, denoted by $\mathbb{C}[G]'$ for the moment: the vector space still equals $V \overset{def}{=} \{e_t\}_{t \in G}$ and

$$\rho'_s : V \to V$$

$$e_u \mapsto \rho'_s(e_u) \overset{def}{=} e_{us}^{-1}.$$ 

The inverse is there to ensure that multiplication is taken in the right order:

$$\rho'_s(e_u) = e_{u(st)}^{-1} = e_{us}^{-1}s^{-1} = \rho'_s(e_{us}) = \rho'_s(\rho'_s(e_u)).$$

Since we will use the left regular representation almost exclusively, the term “regular representation” stands for the left regular representation. Note that the left regular representation and the right regular representation are isomorphic since inversion in $G$ is $G$-linear: for $s, t \in G$, if $\Phi : \mathbb{C}[G] \to \mathbb{C}[G]'$ sends $e_t$ to $e_{t^{-1}}$ and is extended by $\mathbb{C}$-linearity, then

$$\Phi(\rho_s(e_t)) = \Phi(e_{st}) = e_{(st)}^{-1} = e_{st}^{-1}s^{-1} = \rho'_s(e_{s^{-1}}) = \rho'_s(\rho_s(e_t)).$$

- More generally, suppose $G$ acts on the left on a finite set $X$ (we write $G \odot X$ to denote the action), i.e. for any $s \in G$, there is a map $x \mapsto s \cdot x \in X$ such that

$$\forall x \in X, \quad 1 \cdot x = x$$

$$\forall s, t \in G, \quad \forall x \in X, \quad s \cdot (t \cdot x) = (st) \cdot x.$$
Chapter 1

Let $V$ be the free vector space on the set $X$ with basis $\{e_x\}_{x \in X}$ where $e_x : X \to \mathbb{C}$ is the function defined by

$$
\forall y \in X, \quad e_x(y) \overset{\text{def}}{=} \begin{cases} 
1 & \text{if } x = y \\
0 & \text{otherwise.}
\end{cases}
$$

For any $s \in G$, define a map

$$
\lambda_s : V \to V \\
e_x \mapsto e_{sx}.
$$

The induced representation $\lambda : G \to \text{GL}(V)$ is called a permutation representation associated to the group action $G \circlearrowleft X$. Therefore, each left action of $G$ on a finite set $X$ induces a permutation representation of $G$. The same can be done for right actions too: if $X \circlearrowright G$ is a right action of $G$ on the set $X$, for any $s \in G$, define

$$
\rho_s : V \to V \\
e_x \mapsto e_{xs^{-1}}.
$$

Again, the inverse is present to ensure multiplication is taken in the right order. This is also called a permutation representation associated to the (right) action $X \circlearrowright G$.

The most basic example of a permutation representation is given by $X = \{1, \cdots, n\}$ and by taking $G$ to be the group of bijections of $X$ onto itself, called the symmetric group on $n$ letters (or simply the symmetric group), denoted by $S_n$. The associated permutation representation $V = \langle e_1, \cdots, e_n \rangle_\mathbb{C}$ is such that for $\sigma \in S_n$ and $i \in \{1, \cdots, n\}$, we have $\sigma \cdot e_i \overset{\text{def}}{=} e_{\sigma(i)}$. We call this representation the classical representation of $S_n$.

The construction of the free complex vector space on the set $G$ is usually given as follows: as a set, $\langle G \rangle_\mathbb{C} \overset{\text{def}}{=} \text{Hom}_{\text{Set}}(G, \mathbb{C})$, i.e. $\langle G \rangle_\mathbb{C}$ is the set of all functions $f : G \to \mathbb{C}$, which is naturally a complex vector space under pointwise addition and multiplication. It admits a basis of the form $\{e_s | s \in G\}$ such that

$$
e_s(t) \overset{\text{def}}{=} \begin{cases} 
1 & \text{if } s = t \\
0 & \text{otherwise.}
\end{cases}
$$

Under this construction, the left regular representation takes the following form: for $f \in \langle G \rangle_\mathbb{C}$ and $s, x \in G$,

$$
\lambda_s(f)(x) \overset{\text{def}}{=} f(s^{-1}x).
$$

Indeed, for $s, t, x \in G$, $\rho_s(e_t)(x) = e_t(s^{-1}x)$ equals 1 if and only if $s^{-1}x = t$, i.e. if and only if $st = x$, which means that $\rho_s(e_t) = e_{st}$. Therefore, this is exactly the same representation of $G$ but written under a different notation. Analogously, the right regular representation takes the form

$$
\rho_s(f)(x) \overset{\text{def}}{=} f(xs).
$$

This also works for permutation representations associated to left/right actions: if $G \circlearrowleft X$, then $\lambda_s(f)(x) \overset{\text{def}}{=} f(s^{-1}x)$, and if $X \circlearrowright G$, then $\rho_s(f)(x) \overset{\text{def}}{=} f(xs)$. Note that the left (resp. right) regular representation is just the permutation representation of the left (resp. right) action of $G$ on itself, namely $s \cdot t \overset{\text{def}}{=} st$ (resp. $t \cdot s \overset{\text{def}}{=} ts$); the left-hand side is the action and the right-hand side corresponds to multiplication in $G$.

Finally, another way to see that the two notations agree is to see what happens when we write a function $f : X \to \mathbb{C}$ over the basis $\{e_x\}_{x \in X}$ for $V$: for $s \in G$, $f \in V$, write $f = \sum_{x \in X} a_x e_x$ where $a_x = f(x) \in \mathbb{C}$, so that for $y \in X$,

$$
\rho_s(f) = \rho_s \left( \sum_{x \in X} a_x e_x \right) = \sum_{x \in X} a_x \rho_s(e_x) = \sum_{x \in X} a_x e_{sx} = \sum_{x \in X} a_{s^{-1}x} e_x \implies \rho_s(f)(y) = f(s^{-1}y).
$$
Theorem 1.6. Let $G$ be a group. The collection of all complex representations of $G$ (which we assume finite-dimensional) together with $G$-linear maps as morphisms form a category; we denote it by $\text{Rep}_C(G)$. For reasons explained later (c.f. Chapter 3), the hom-set between two representations of $G$, say $V$ and $W$, is denoted by $\text{Hom}_{\text{Rep}_C(G)}(V, W)$ in this category.

Proof. It is clear that the identity map $\text{id}_V$ of the vector space $V$ of a complex representation $(V, \rho)$ of $G$ is $G$-linear. Given two $G$-linear maps $\tau_1 : (V_1, \rho^1) \to (V_2, \rho^2)$ and $\tau_2 : (V_2, \rho^2) \to (V_3, \rho^3)$, their composition $\tau_2 \circ \tau_1 : (V_1, \rho^1) \to (V_3, \rho^3)$ is a $G$-linear map since the following diagram commutes for any $s \in G$.

\[
\begin{array}{c}
V_1 \xrightarrow{\tau_1} V_2 \xrightarrow{\tau_2} V_3 \\
\downarrow{\rho^1_s} & \downarrow{\rho^2_s} & \downarrow{\rho^3_s} \\
V_1 \xrightarrow{\tau_1} V_2 \xrightarrow{\tau_2} V_3
\end{array}
\]

1.2 Subrepresentations

Definition 1.7. Let $\rho : G \to \text{GL}(V)$ be a linear representation of $G$ and let $W \subseteq V$ be a vector subspace. We say that $W$ is $G$-stable under the action of $G$ if

\[ \forall w \in W, \forall s \in G, \quad \rho_s(w) \in W. \]

If $W \subseteq V$ is $G$-stable, the restriction

\[ \rho^W_s \overset{\text{def}}{=} \rho_s|_W : W \to W \]

is an isomorphism and $\rho^W_{s} = \rho^W_s \circ \rho^W_t$. It follows that $\rho^W : G \to \text{GL}(W)$ is a representation of $G$, called a subrepresentation of $V$.

Example 1.8. Let $V$ be the regular representation of $G$ and write

\[ W = \langle x \rangle, \quad x \overset{\text{def}}{=} \sum_{s \in G} e_s. \]

We have

\[ \rho_t(x) = \sum_{s \in G} \rho_t(e_s) = \sum_{s \in G} e_{ts} = x, \]

hence $W$ is a $G$-stable subspace of $V$. Therefore, we conclude that $W$ is a subrepresentation of $V$ isomorphic to the trivial representation (as $\rho_s(x) = x$ for any $s \in G$).

Definition 1.9. Assume $W, W'$ are two vector subspaces of the complex vector space $V$. We write $V = W \oplus W'$ if and only if $V = W + W'$ and $W \cap W' = \{0\}$. In other words, any element $v \in V$ can be written uniquely as $v = w + w'$ where $w \in W$ and $w' \in W'$. In particular, $\dim V = \dim W + \dim W'$. We call $W'$ a complement of $W$ in $V$. A projection of $V$ is a linear map $p : V \to V$ satisfying $p \circ p = p$; the condition is equivalent to $p(w) = w$ for all $w \in \text{im} p$.

Remark 1.10. If $V = W \oplus W'$, then we have a linear projection $p : V \to V$ with image $W$ given by $v = w + w' \mapsto w$. Conversely, given a linear projection $p : V \to V$ where $W \overset{\text{def}}{=} \text{im} p \subseteq V$, we see that $V = W \oplus \ker p$ (write $x \in V$ as $x = p(x) + (x - p(x)) \in W + \ker f$; clearly $W \cap \ker p = 0$). Therefore, we have a bijection

\[ \{\text{linear projections } p : V \to V \text{ with image } W\} \leftrightarrow \{\text{complements } W' \text{ of } W \text{ in } V\}. \]
Lemma 1.11. Let $V$ be a linear representation of $G$ and $W, W' \subseteq V$ two linear subspaces such that $V = W \oplus W'$. These come with two linear projections $p, p' : V \rightarrow V$ with

$$\ker p = W', \quad \text{im } p = W, \quad \ker p' = W, \quad \text{im } p' = W'.$$

The following are equivalent:

(i) The map $p$ is $G$-linear.

(ii) The map $p'$ is $G$-linear.

(iii) The subspaces $W$ and $W'$ are $G$-stable subspaces of $V$.

**Proof.** (i) $\Rightarrow$ (iii) Let $x \in W' = \ker p$. For $s \in G$, we have $p(\rho_s(x)) = \rho_s(p(x)) = \rho_s(0) = 0$, thus $\rho_s(w) \in \ker p$ and $\ker p$ is $G$-stable. For $w \in W$, we have

$$\rho_s(w) = \rho_s(p(w)) = p(\rho_s(w)) \in W,$$

showing that $W$ is also $G$-stable.

(iii) $\iff$ (i) If $v \in V$ and $s \in G$, writing $v = w + w'$ with $w \in W, w' \in W'$ gives

$$\rho_s(p(v)) = \rho_s(w) = p(\rho_s(w)) + p(\rho_s(w')) = p(\rho_s(v)).$$

(ii) $\iff$ (iii) Simply reverse the roles of $p$ and $p'$ and use (i) $\iff$ (iii).

Theorem 1.12. (Maschke’s Theorem) Let $\rho : G \rightarrow GL(V)$ be a linear representation of $G$ and $W \subseteq V$ a $G$-stable subspace. Then there exists a $G$-stable vector subspace $W' \subseteq V$ for which $V = W \oplus W'$.

**Proof.** Let $W'$ be an arbitrary complement of $W$ in $V$ and let $p : V \rightarrow V$ be the corresponding linear projection (i.e. $\text{im } p = W$ and $\ker p = W'$). Set $g \overset{\text{def}}{=} \text{ord}(G)$. Consider the following linear map (a so-called “average of $p$ over $G$”):

$$\bar{p} \overset{\text{def}}{=} \frac{1}{g} \sum_{t \in G} \rho_t \circ p \circ \rho_t^{-1}.$$

Moreover, since $\rho_t^{-1}(w) = \rho_{t^{-1}}(w) \in W$ for all $w \in W$, we notice that

$$p(\rho_t^{-1}(w)) = \rho_t^{-1}(w) \implies (\rho_t \circ p \circ \rho_t^{-1})(w) = w \implies \bar{p}(w) = w.$$

Since $W$ is $G$-stable, $\bar{p}$ maps $V$ onto $W$ (because $p$ does), so this means $\bar{p} : V \rightarrow V$ is a projection with image $W$. For any $s \in G$,

$$\rho_s \circ \bar{p} \circ \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_s \circ \rho_t \circ p \circ \rho_t^{-1} \circ \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} \rho_{st} \circ p \circ \rho_{(st)^{-1}}^{-1} = \bar{p},$$

hence $\bar{p}$ is a $G$-linear projection. By Lemma 1.11, we see that $W' \overset{\text{def}}{=} \ker \bar{p}$ is $G$-stable and $V = W \oplus W'$.

Remark 1.13. One sees in the proof of Theorem 1.12 that $\bar{p} = \bar{p}$, so that the construction $p \mapsto \bar{p}$ projects the set of linear projections $p : V \rightarrow V$ onto those that are $G$-linear. This is possible because we can average over $G$, i.e. because $G$ is a finite group and $g \in \mathbb{C}^\times$ (when either of these conditions is removed, say by changing the field or working over infinite groups, the situation becomes much more complicated). Because it will prove so useful, Maschke’s theorem is one of the main reasons we restrict our attention to finite groups.
Remark 1.14. Let \( G \) be a finite group, \( g \equiv \text{ord}(G) \), let \( V \) be a complex vector space and denote (in the entire document) the complex conjugation map by a bar, i.e. \( \overline{\cdot} : \mathbb{C} \to \mathbb{C} \). A hermitian inner product on \( V \) (or simply inner product) is a map \((\cdot | \cdot) : V \times V \to \mathbb{C}\) satisfying the three following properties:

(i) **Conjugate symmetry**: for all \( x, y \in V \), \((x | y) = (\overline{y} | x)\)

(ii) **Sesquilinearity**: for all \( x, y, z, w \in V \) and \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \),

\[
(\alpha x + \beta y | z) = \alpha (x | z) + \beta (y | z), \quad (x | \gamma z + \delta w) = \overline{\gamma} (x | z) + \overline{\delta} (x | w)
\]

(iii) **Positive-definiteness**: for all \( x \in V \), \((x | x) \geq 0\) and \((x | x) = 0\) if and only if \( x = 0\).

A pair \((V, (\cdot | \cdot))\) is called an inner product space. Given a basis for \( V \), an isomorphism \( V \cong \mathbb{C}^n \) induces an inner product space on \( V \) via the standard inner product on \( \mathbb{C}^n \), namely

\[
\left( \sum_{i=1}^{n} x_i e_i \right) \left( \sum_{i=1}^{n} y_i e_i \right) \equiv \sum_{i=1}^{n} x_i \overline{y}_i.
\]

Therefore, any complex vector space can be equipped with an inner product space.

When \( W, W' \leq V \) are linear subspaces and \( v \in V \), we say that \( W \) and \( W' \) are orthogonal (which we denote by \( W \perp W' \)) if for any \( w \in W \), \( w' \in W' \), we have \((w | w') = 0\); if \( W' = \{v\}_\mathbb{C} \) for some \( v \in V \), we also write \( v \perp W \). Given an inner product on a linear representation \( V \), one says that it is \( G \)-invariant if it satisfies

\[
\forall s \in G, \quad \forall x, y \in V, \quad (\rho_s(x) | \rho_s(y)) = (x | y).
\]

A \( G \)-invariant inner product \( (\cdot | \cdot)' \) can always be constructed out of an inner product \( (\cdot | \cdot) \) by letting

\[
(\cdot | \cdot) \mapsto (\cdot | \cdot)' \equiv \frac{1}{g} \sum_{s \in G} (\rho_s(\cdot) | \rho_s(\cdot))
\]

which maps an inner product to an invariant one. Note that if \((\cdot | \cdot)\) is \( G \)-invariant, then this construction does not change the \( G \)-invariant inner product, i.e. \((\cdot | \cdot)' = (\cdot | \cdot)\). We deduce that any complex representation admits a \( G \)-invariant inner product. The orthogonal complement of \( W \) in \( V \) is defined as

\[
W^\perp \equiv \{ v \in V \mid v \perp W \}.
\]

Under such an inner product, the orthogonal complement \( W^\perp \) of a \( G \)-stable subspace \( W \) of \( V \) is also a \( G \)-stable subspace since for \( w^\perp \in W^\perp \) and all \( w \in W \), we have

\[
(w | \rho_s(w^\perp)) = (\rho_s^{-1}(w) | w^\perp) = 0 \implies \rho_s(w^\perp) \perp W.
\]

1.3 Operations on representations

Definition 1.15. Keeping the notations of Theorem 1.12, \( V \) is a linear representation of \( G \); \( W, W' \) are \( G \)-stable subspaces with \( V = W \oplus W' \) as linear subspaces so that \( v \in V \) has a unique expression of the form \( v = w + w' \) where \( w \in W \), \( w' \in W' \). It follows that

\[
\rho_s(v) = \sum_{w \in W} \rho_s(w) + \rho_s(w')
\]

so that the subrepresentations \( W \) and \( W' \) determine the representation \( V \).
We say that a finite-dimensional vector space $V$ is the direct sum of the two representations $W$ and $W'$ when $V = W \oplus W'$ and $W, W'$ are $G$-stable subspaces. If the representations $W$ and $W'$ are given in matrix form by the matrices $R_s$ and $R'_s$, then the representation $W \oplus W'$ is given in matrix form by

$$\rho_s^V \implies \begin{bmatrix} R_s & 0 \\ 0 & R'_s \end{bmatrix}$$

(using block form notation). If $W$ and $W'$ were to be linear subspaces of $V$ for which $V = W \oplus W'$ only as vector spaces and not as subrepresentations, we would explicitly mention it so that the notation $V = W \oplus W'$ for representations automatically stands for the direct sum of subrepresentations (c.f. Definition 2.65 for instance).

**Remark 1.16.** Recall that the tensor product of two complex vector spaces $V_1, V_2$ is the vector space $V_1 \otimes V_2$ together with a $\mathbb{C}$-bilinear map $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$ (written $(v_1, v_2) \mapsto v_1 \otimes v_2$) satisfying the following property: for any $\mathbb{C}$-bilinear map $L : V_1 \times V_2 \to W$, there exists a unique linear map $\tilde{L} : V_1 \otimes V_2 \to W$ making the following diagram commute:

$$V_1 \times V_2 \overset{\otimes}{\longrightarrow} V_1 \otimes V_2 \overset{L}{\longrightarrow} W \overset{\tilde{L}}{\longrightarrow} W$$

or in other words, $L(v_1, v_2) = \tilde{L}(v_1 \otimes v_2)$. In particular, this tells us how to define linear maps $V_1 \otimes V_2 \to W$, namely through $\mathbb{C}$-bilinear maps $V_1 \times V_2 \to W$.

In category-theoretical terms, given complex vector spaces $V_1, V_2$, the construction which sends a vector space $W$ to the set of $\mathbb{C}$-bilinear maps $\text{Bil}(V_1, V_2; W)$ is functorial; given a linear map $W \to W'$, post-composition gives a map $\text{Bil}(V_1, V_2; W) \to \text{Bil}(V_1, V_2; W')$. The corresponding functor is represented by $V_1 \otimes V_2$ by the above commutative diagram, which gives us the following natural bijection

$$\text{Bil}(V_1, V_2; W) \simeq \text{Hom}_\mathbb{C}(V_1 \otimes V_2, W).$$

**Definition 1.17.** A linear representation $\rho : G \to \text{GL}(V)$ induces a natural left group action $G \circ V$ for $s \in G$ and $v \in V$, we set $s \cdot v \overset{\text{def}}{=} \rho_s(v)$. We call this the canonical $G$-action of the representation. A $G$-linear map $\varphi : V \to W$ is therefore equivalent to a $\mathbb{C}$-linear map which commutes with the action of $G$, namely $s \cdot \varphi(v) = \varphi(s \cdot v)$ for all $s \in G$ and $v \in V$. In particular, defining a representation $\rho : G \to \text{GL}(V)$ is the same as defining an action $G \circ V$ where $G$ acts by $\mathbb{C}$-linear endomorphisms of $V$, which we call a $G$-linear action on $V$.

**Definition 1.18.** Let $V_1, V_2$ be representations of the finite group $G$. Define a representation of $G$ over $V_1 \otimes V_2$ as follows: for $s \in G$,

$$s \cdot (v_1 \otimes v_2) \overset{\text{def}}{=} (s \cdot v_1) \otimes (s \cdot v_2).$$

This gives us a representation $\rho : G \to \text{GL}(V_1 \otimes V_2)$ called the tensor product of the two representations $\rho^1$ and $\rho^2$ because each $s$ induces the $\mathbb{C}$-bilinear map $(v_1, v_2) \mapsto (s \cdot v_1, s \cdot v_2)$, so that the above defined map is a $\mathbb{C}$-linear isomorphism of $V_1 \otimes V_2$. We denote the corresponding morphism of groups by $\rho \overset{\text{def}}{=} \rho^1 \otimes \rho^2$.

**Remark 1.19.** If we write $\rho^i$ in matrix form over the bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$, we get matrices $(r_{i,j}(s))$ and $(r_{i,j'}(s))$ for each $s \in G$. The matrix form of $\rho \overset{\text{def}}{=} \rho^1 \otimes \rho^2$ is computed as follows: since
\{e_i \otimes f_j\} is a basis for \(V_1 \otimes V_2\),
\[
\rho(e_{i_1} \otimes f_{j_2}) = \rho^1(e_{i_1}) \otimes \rho^2(f_{j_2}) = \left( \sum_{i_1=1}^n r_{i_1j_1}(s)e_{i_1} \right) \otimes \left( \sum_{i_2=1}^m r_{i_2j_2}(s)f_{i_2} \right) = \sum_{i_1=1}^n \sum_{i_2=1}^m (r_{i_1j_1}(s)r_{i_2j_2}(s))e_{i_1} \otimes f_{i_2}.
\]

It follows that the \(((i_1,i_2),(j_1,j_2))\)-coefficient of the matrix representation of \(\rho_1 \otimes \rho_2\) is \(r_{i_1j_1}(s)r_{i_2j_2}(s)\).

**Proposition 1.20.** Let \(V_1, V_2, W\) be two representations of \(G\). The group \(G\) acts on \(V_1 \times V_2\) diagonally, namely \(s \cdot (v_1, v_2) \overset{\text{def}}{=} (s \cdot v_1, s \cdot v_2)\). A \(\mathbb{C}\)-bilinear map \(\varphi : V_1 \times V_2 \to W\) is said to **preserve the action of** \(G\) if for any \(s \in G, v_1, v_2 \in V\), we have
\[
\varphi(s \cdot (v_1, v_2)) = s \cdot \varphi(v_1, v_2).
\]

The tensor product representation \(V_1 \otimes V_2\) is characterized by the following universal property. By Remark 1.16, a \(\mathbb{C}\)-bilinear map \(\varphi : V_1 \times V_2 \to W\) corresponds to a \(\mathbb{C}\)-linear map \(\tilde{\varphi} : V_1 \otimes V_2 \to W\). The tensor product representation \(\rho : G \to \text{GL}(V_1 \otimes V_2)\) is such that \(\varphi\) respects the action of \(G\) if and only if \(\tilde{\varphi}\) is a \(G\)-linear map. Just as the tensor product of vector spaces, the tensor product representation is unique up to a unique \(G\)-linear isomorphism respecting \(\otimes\).

**Proof.** We have a functor \(\text{Rep}_G(G) \to \text{Set}\) which maps a representation \(W\) to the set \(\text{Bil}_G(V_1, V_2; W)\) of \(\mathbb{C}\)-bilinear maps \(V_1 \times V_2 \to W\) which respect the action of \(G\). It suffices to show that this functor is represented by \(V_1 \otimes V_2\) with the given \(G\)-linear action. Remark 1.16 shows that the functor \(\text{Bil}(V_1, V_2; -)\) is already represented by the vector space \(V_1 \otimes V_2\). To see that the tensor product representation represents \(\text{Bil}_G(V_1, V_2; W)\), it suffices to show that the bijection of Remark 1.16 maps \(\text{Bil}_G(V_1, V_2; W)\) bijectively onto \(\text{Hom}_{\mathbb{C}[G]}(V_1 \otimes V_2, W)\). If \(\varphi : V_1 \times V_2 \to W\) respects the action of \(G\), then for \(s \in G\),
\[
\tilde{\varphi}(s \cdot (v_1 \otimes v_2)) = \tilde{\varphi}(s \cdot v_1) \otimes (s \cdot v_2) = \varphi(s \cdot v_1, s \cdot v_2) = s \cdot \varphi(v_1, v_2) = s \cdot \tilde{\varphi}(v_1 \otimes v_2).
\]

Conversely, if \(\tilde{\varphi} : V_1 \otimes V_2 \to W\) is \(G\)-linear, then
\[
\varphi(s \cdot v_1, s \cdot v_2) = \tilde{\varphi}(s \cdot v_1) \otimes (s \cdot v_2) = \tilde{\varphi}(s \cdot (v_1 \otimes v_2)) = s \cdot \tilde{\varphi}(v_1 \otimes v_2) = s \cdot \varphi(v_1, v_2).
\]

**Example 1.21.** Consider a vector space \(V\) and the tensor product \(W = V \otimes V\) together with a representation \(\rho : G \to \text{GL}(V)\). The \(\mathbb{C}\)-bilinear map \(V \times V \to V \times V\) defined by \((v_1, v_2) \mapsto (v_2, v_1)\) induces an isomorphism \(\theta : V \otimes V \to V \otimes V\) which is characterized by the formula \(\theta(v_1 \otimes v_2) = v_2 \otimes v_1\) for any \(v_1, v_2 \in V\); in particular, \(\theta \circ \theta = \text{id}_{V \otimes V}\). There are two relevant subspaces of \(V \otimes V\) worth mentioning, namely
\[
\text{Sym}^2(V) \overset{\text{def}}{=} \{z \in V \otimes V \mid \theta(z) = z\}, \quad \text{Alt}^2(V) = \{z \in V \otimes V \mid \theta(z) = -z\}.
\]

It is not hard to see that \(V \otimes V = \text{Sym}^2(V) \oplus \text{Alt}^2(V)\) since for all \(\alpha \in V \otimes V\), we have
\[
\alpha = \frac{\alpha + \theta(\alpha)}{2} + \frac{\alpha - \theta(\alpha)}{2} \in \text{Sym}^2(V) \cup \text{Alt}^2(V)
\]
and if \(z \in \text{Sym}^2(V) \cap \text{Alt}^2(V)\), then \(z = \theta(z) = -z\), hence \(z = 0\). The two subspaces \(\text{Sym}^2(V), \text{Alt}^2(V)\) come with projections \(p_{\text{Sym}} : V \otimes V \to V \otimes V\) and \(p_{\text{Alt}} : V \otimes V \to V \otimes V\) defined by the above formulas,
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namely for $\alpha \in V \otimes V$,

$$p_{\text{Sym}}(\alpha) \overset{\text{def}}{=} \frac{\alpha + \theta(\alpha)}{2}, \quad p_{\text{Alt}}(\alpha) \overset{\text{def}}{=} \frac{\alpha - \theta(\alpha)}{2}.$$  

Clearly,

$$\ker p_{\text{Sym}} = \text{Alt}^2(V), \quad \im p_{\text{Sym}} = \text{Sym}^2(V), \quad \ker p_{\text{Alt}} = \text{Sym}^2(V), \quad \im p_{\text{Alt}} = \text{Alt}^2(V).$$

With respect to the tensor product representation $\rho \otimes \rho : G \to \text{GL}(V \otimes V)$, since $\theta$ and $\text{id}_{V \otimes V}$ are both $G$-linear endomorphisms of $V \otimes V$, so are $p_{\text{Sym}}$ and $p_{\text{Alt}}$, which means that $\text{Sym}^2(V)$ and $\text{Alt}^2(V)$ are $G$-stable subspaces by Lemma 1.11, e.g. subrepresentations of $V \otimes V$. We call them the **symmetric square** and **exterior square** of $V$, respectively.

Constructions of representations where permutations are involved will be studied in more detail in subsequent sections when we investigate properties of the symmetric group $S_d$ for $d \geq 2$.

**Definition 1.22.** Let $\rho^i : G_i \to \text{GL}(V_i)$ be a representation of the finite group $G_i$ for $i = 1, 2$. Define a representation $\rho^1 \boxtimes \rho^2 : G_1 \times G_2 \to \text{GL}(V_1 \boxtimes V_2)$, called the **box product** of $\rho^1$ and $\rho^2$. As vector spaces, $V_1 \boxtimes V_2 \overset{\text{def}}{=} V_1 \otimes V_2$ and we denote simple tensors in $V_1 \boxtimes V_2$ by $v_1 \boxtimes v_2$. For $(s_1, s_2) \in G_1 \times G_2$,

$$(\rho^1 \boxtimes \rho^2)(s_1, s_2)(v_1 \boxtimes v_2) \overset{\text{def}}{=} \rho^1_{s_1}(v_1) \boxtimes \rho^2_{s_2}(v_2).$$

**Remark 1.23.** The box product of representations and the tensor product of representations are related operations. If $G_1 = G_2 = G$, $\rho^i : G \to \text{GL}(V_i)$ are two representations for $i = 1, 2$ and $\Delta : G \to G \times G$ denotes the diagonal morphism (i.e. $\Delta(s) \overset{\text{def}}{=} (s, s)$), then $\rho^1 \otimes \rho^2 = (\rho^1 \boxtimes \rho^2) \circ \Delta$.

Note that in the literature, both operations ($\boxtimes$ and $\otimes$) are usually called the tensor product, but we wished to distinguish such operations because even though they are related, they do not live over the same groups.

**Proposition 1.24.** Let $\rho^i : G_i \to \text{GL}(V_i)$ be representations, $i = 1, 2$. Given a representation $W$ of the group $G_1 \times G_2$, a $\mathbb{C}$-bilinear map $\varphi : V_1 \times V_2 \to W$ is said to respect the action of $G_1 \times G_2$ if and only for any $(s_1, s_2) \in G_1 \times G_2$ and $(v_1, v_2) \in V_1 \times V_2$, we have $\varphi(s_1 \cdot v_1, s_2 \cdot v_2) = (s_1, s_2) \cdot \varphi(v_1, v_2)$. The box product representation $\rho^1 \boxtimes \rho^2 : G_1 \times G_2 \to \text{GL}(V_1 \boxtimes V_2)$ satisfies the following universal property: a $\mathbb{C}$-bilinear map $\varphi : V_1 \times V_2 \to W$ respects the action of $G_1 \times G_2$ if and only if the corresponding map $\tilde{\varphi} : V_1 \boxtimes V_2 \to W$ is $G_1 \times G_2$-linear.

**Proof.** The proof is entirely analogous to Proposition 1.20; all we need to show is that the natural bijection $\text{Bil}(V_1, V_2; W) \simeq \text{Hom}_\mathbb{C}(V_1 \otimes V_2, W)$ establishes a correspondence between the subset $\text{Bil}_{G_1,G_2}(V_1, V_2; W)$ of bilinear maps which respects the action of $G_1 \times G_2$ (which is a functorial construction in $W$, e.g. as a functor from $\text{Rep}_\mathbb{C}(G_1 \times G_2)$ to $\text{Set}$) and $\text{Hom}_{\mathbb{C}[G_1 \times G_2]}(V_1 \boxtimes V_2, W)$. To proceed, we need to show that the bijection of Remark 1.16 maps one subset into the other. If $\varphi : V_1 \times V_2 \to W$ respects the action of $G_1 \times G_2$, then for $(s_1, s_2) \in G_1 \times G_2$ and $(v_1, v_2) \in V_1 \times V_2$, we have

$$\tilde{\varphi}((s_1, s_2) \cdot (v_1 \boxtimes v_2)) = \varphi(s_1 \cdot v_1, s_2 \cdot v_2) = (s_1, s_2) \cdot \varphi(v_1, v_2) = (s_1, s_2) \cdot \varphi(v_1 \boxtimes v_2).$$

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Conversely, if $\tilde{\varphi} : V_1 \boxtimes V_2 \to W$ is $G_1 \times G_2$-linear, then

\[
\varphi(s_1 \cdot v_1, s_2 \cdot v_2) = \tilde{\varphi}((s_1 \cdot v_1) \boxtimes (s_2 \cdot v_2)) = \tilde{\varphi}((s_1, s_2) \cdot (v_1 \boxtimes v_2)) = (s_1, s_2) \cdot \tilde{\varphi}(v_1 \boxtimes v_2) = (s_1, s_2) \cdot \varphi(v_1, v_2).
\]

### 1.4 Irreducible representations

**Definition 1.25.** Let $\rho : G \to \text{GL}(V)$ be a linear representation of $G$. We say that $V$ is **irreducible** if $V \neq 0$ and the set of subrepresentations of $V$ equals $\{0, V\}$.

**Remark 1.26.**

- By Maschke’s Theorem, $V$ is irreducible if and only if $V$ is not the direct sum of two proper subrepresentations (i.e. the only options are $V = V \oplus 0$).

- Every representation of degree 1 is irreducible. We will see later (c.f. Theorem 2.53) that any finite non-abelian group admits an irreducible representation of degree greater than 1, which separates the representation theory of abelian groups from that of non-abelian groups.

**Theorem 1.27.** Every representation of a finite group $G$ is a direct sum of irreducible subrepresentations.

**Proof.** If $\dim V = 0$, the representation is the empty direct sum. We proceed by induction on $\dim V \geq 1$. If $\dim V = 1$, then $V$ is irreducible, so we are done. Assume $\dim V > 1$. If $V$ is irreducible, we are again done; assume not, so that $V$ is not irreducible. By Maschke’s Theorem, there exists a decomposition $V = V' \oplus V''$ with $\dim V', \dim V'' < \dim V$. Since $V'$ and $V''$ can be decomposed as a direct sum of irreducible representations, so does $V$.

**Remark 1.28.** In general, this decomposition is not unique. For instance, if $\rho_s = \text{id}_{V'}$ for all $s \in G$, then any subspace of $V$ is a subrepresentation, so for any basis $\{v_1, \ldots, v_n\}$ of $V$ where $n \overset{\text{def}}{=} \dim V$, we have

\[
V = \bigoplus_{i=1}^{n} (v_i)_{\mathbb{C}},
\]

so we have as many possible decompositions for $V$ as we have bases (modulo scalars). However, what is preserved in any decomposition is the number of irreducible subrepresentations appearing which are isomorphic to a given irreducible representation $W$ (for instance, in the above example, the trivial representation appears in $V$ precisely $n$ times as a subrepresentation, regardless of the chosen basis). This will be shown as a consequence of Schur’s Lemma (c.f. Theorem 1.30).

**Lemma 1.29.** Let $\rho^1 : G \to \text{GL}(V_1)$ and $\rho^2 : G \to \text{GL}(V_2)$ be two representations of $G$. Then the set

\[
\text{Hom}_{\mathbb{C}[G]}(V_1, V_2) \overset{\text{def}}{=} \{ f \in \text{Hom}_{\mathbb{C}}(V_1, V_2) \mid \forall s \in G, \ f \circ \rho^1_s = \rho^2_s \circ f \}
\]

is a complex vector subspace of $\text{Hom}_{\mathbb{C}}(V_1, V_2)$.

**Proof.** One can see $\text{Hom}_{\mathbb{C}[G]}(V_1, V_2)$ as the image of the linear projection

\[
\Phi : \text{Hom}_{\mathbb{C}}(V_1, V_2) \to \text{Hom}_{\mathbb{C}}(V_1, V_2), \ f \mapsto \frac{1}{g} \sum_{s \in G} \rho^2_s \circ f \circ \rho^1_s^{-1},
\]
so that \( \text{Hom}_\mathbb{C}[G](V_1, V_2) = \Phi(\text{Hom}_\mathbb{C}(V_1, V_2)) \) is a linear subspace of \( \text{Hom}_\mathbb{C}(V_1, V_2) \).

**Theorem 1.30.** (Schur’s Lemma) Let \( \rho^i : G \to \text{GL}(V_i) \) be irreducible representations of the finite group \( G \) and \( f : V_1 \to V_2 \) be a \( G \)-linear map.

(i) The kernel \( \ker f \) is a subrepresentation of \( V_1 \) and the image \( \text{im} f \) is a subrepresentation of \( V_2 \) (this does not assume \( V_1 \) or \( V_2 \) irreducible). In particular, either \( \rho^1 \) is isomorphic to \( \rho^2 \) or \( f = 0 \).

(ii) If \( \rho^1 \) is isomorphic to \( \rho^2 \), let \( \theta : V_1 \to V_2 \) be a \( G \)-linear isomorphism. We get \( f = \lambda \theta \) for a uniquely determined \( \lambda \in \mathbb{C} \).

**Proof.** (i) The subspace \( \ker f \subseteq V_1 \) is \( G \)-stable, since if \( v_1 \in \ker f \), then for \( s \in G \),

\[
 f(s \cdot v_1) = s \cdot f(v_1) = s \cdot 0 = 0 \implies s \cdot v_1 \in \ker f.
\]

Since \( \ker f \) is \( G \)-stable and \( V_1 \) is irreducible, \( \ker f \subseteq \{0, V_1\} \).

Similarly, if \( v_2 = f(v_1) \in \text{im} f \), then for \( s \in G \), we have

\[
 s \cdot v_2 = s \cdot f(v_1) = f(s \cdot v_1) \in \text{im} f.
\]

It follows that \( \text{im} f \) is \( G \)-stable; since \( V_2 \) is irreducible, \( \text{im} f \subseteq \{0, V_2\} \).

If \( \ker f = 0 \), we cannot have \( \text{im} f = 0 \) because \( 0 \neq f(V_1) \subseteq V_2 \), hence \( \text{im} f = V_2 \); if \( \ker f = V_1 \), we must have \( \text{im} f = 0 \). Therefore \( f = 0 \) or \( f \) is an isomorphism between \( \rho^1 \) and \( \rho^2 \), which ends the proof.

(ii) Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( \theta^{-1} \circ f \), so that \( (\theta^{-1} \circ f)(v) = \lambda v \) for some \( v \in V_1 \setminus \{0\} \), or in other words, \( f(v) = \lambda \theta(v) \). Set \( f' \overset{\text{def}}{=} f - \lambda \theta \), e.g. \( f' : V_1 \to V_2 \). By Lemma 1.29, we can apply part (i) to \( f' \) so that \( \ker f' = V_1 \) because \( v \in \ker f' \neq 0 \) and \( V_1 \) is irreducible. This implies \( f' = 0 \), or in other words, \( f = \lambda \theta \).

**Definition 1.31.** Let \( V, W \) be complex vector spaces and set \( V^\ast \overset{\text{def}}{=} \text{Hom}_\mathbb{C}(V, \mathbb{C}) \), which we call the dual vector space of \( V \); it is the set of \( \mathbb{C} \)-linear maps from \( V \) to \( \mathbb{C} \). There is a natural map

\[
 \Phi : V^\ast \otimes W \to \text{Hom}_\mathbb{C}(V, W), \quad \varphi \otimes w \mapsto (v \mapsto \varphi(v)w).
\]

It is an isomorphism because we assume \( V \) and \( W \) finite-dimensional. To see this, let \( \{v_1, \cdots, v_n\} \) be a basis of \( V \) and \( \{\omega_1, \cdots, \omega_n\} \) be the corresponding dual basis, so that \( \omega_i(v_j) = \delta_{ij} \) where \( \delta_{ij} \) denotes the Kronecker delta. Let \( \{w_1, \cdots, w_m\} \) be a basis of \( W \). It follows that

\[
 \Phi(\omega_j \otimes w_i) \mapsto (v_k \mapsto \omega_j(v_k)w_i = \delta_{jk}w_i)
\]

so that \( \Phi \) maps a basis of \( V^\ast \otimes W \) bijectively to a basis of \( \text{Hom}_\mathbb{C}(V, W) \) (in matrix form, the linear maps \( v_k \mapsto \omega_j(v_k)w_i \) correspond to matrices with zeros everywhere except the 1 at the \( (i, j) \)-coordinate). When \( V = W \), there is also a natural evaluation map

\[
 \text{ev} : V^\ast \otimes V \to \mathbb{C}, \quad \varphi \otimes v \mapsto \varphi(v).
\]

The map \( \text{tr} \overset{\text{def}}{=} \text{ev} \circ \Phi^{-1} \) is called the trace map of \( V \). Given a basis \( \{v_1, \cdots, v_n\} \) of \( V \) and writing a linear map \( f : V \to V \) in matrix form so that

\[
 f(v_j) = \sum_{i=1}^n a_{ij}v_i,
\]

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over the basis \( \{ \omega_j \otimes v_i \mid i, j = 1, \ldots, n \} \) of \( V^* \otimes V \), we have
\[
\Phi \left( \sum_{i,j=1}^{n} a_{ij} \omega_j \otimes v_i \right) = \sum_{i,j=1}^{n} a_{ij} \omega_j(v_i)v_i = \sum_{i=1}^{n} a_{ik}v_i = f(v_k),
\]
hence \( \Phi \left( \sum_{i,j=1}^{n} a_{ij} \omega_j \otimes v_i \right) = f \) and we obtain the formula
\[
\text{tr} (f) = \sum_{i,j=1}^{n} \text{ev}(a_{ij} \omega_j \otimes v_i) = \sum_{i,j=1}^{n} a_{ij} \delta_{ij} = \sum_{i=1}^{n} a_{ii}.
\]

If \( f, g : V \to V \), then \( \text{tr} (f \circ g) = \text{tr} (g \circ f) \). It follows from this fact (or the definition itself, since it does not depend on a choice of basis) that if \( g \) is an automorphism of \( V \), then \( \text{tr} (g \circ f \circ g^{-1}) = \text{tr} (f) \). Also recall that the matrix form of the dual map \( f^* : V^* \to V^* \) defined by sending \( g : V \to \mathbb{C} \) to \( g \circ f \) is the transpose of the matrix of \( f \) when \( V \) and \( V^* \) are given the bases \( \{ v_1, \ldots, v_n \} \) and \( \{ \omega_1, \ldots, \omega_n \} \) respectively, so that \( \text{tr} (f^*) = \text{tr} (f) \). The trace will be used abundantly when we will have developed character theory.

**Remark 1.32.** Let \( V \) be a complex vector space and \( f \in \text{Hom}_\mathbb{C}(V, V) \). If \( \{ v_1, \ldots, v_n \} \) is a basis over which \( f \) has Jordan normal form, meaning that \( f(v_i) \in \{ \lambda_i v_i, \lambda_i v_i + v_{i-1} \} \) for \( i = 1, \ldots, n \), we see by definition that \( \text{tr} (f) \) is the sum of the eigenvalues of \( f \) with algebraic multiplicity.

A vector \( v \in V \setminus \{0\} \) is a **geometric eigenvector** for \( f \) if there exists \( \lambda \in \mathbb{C} \) such that \( f(v) = \lambda v \), in which case we call \( \lambda \) a **geometric eigenvalue** (or simply eigenvalue). An **algebraic eigenvector** is a vector \( v \in V \) in the kernel of the linear map \( (f - \lambda \text{id}_V)^k \) for some \( k \geq 1 \) and \( \lambda \in \mathbb{C} \) (the \( k \)th exponent denotes \( k \)-fold composition). When \( \text{ker}(f - \lambda \text{id}_V)^k \neq 0 \), we call \( \bigcup_{k \geq 0} \text{ker}(f - \lambda \text{id}_V)^k \) the **algebraic eigenspace** of \( f \) associated to \( \lambda \), called an **algebraic eigenvalue**, whereas when \( \text{ker}(f - \lambda \text{id}_V) \neq 0 \), \( \text{ker}(f - \lambda \text{id}_V) \) is called the **geometric eigenspace**. Under these definitions, a complex matrix is diagonalizable if and only if its geometric and algebraic eigenspaces are equal (the statement is obvious in Jordan normal form).

Note that in Jordan normal form, the basis vectors corresponding to a Jordan block are precisely those spanning the algebraic eigenspace of the corresponding algebraic eigenvalue on the diagonal of that block. The algebraic multiplicity of an algebraic eigenvalue is defined as the dimension of its corresponding algebraic eigenspace, so that the trace is easily seen to be the sum of the algebraic eigenvalues, counted with algebraic multiplicity.

**Corollary 1.33.** Let \( V_1, V_2 \) be two irreducible representations of \( G \) and \( h : V_1 \to V_2 \) be a linear map. Set \( g \overset{\text{def}}{=} \text{ord}(G) \). Define
\[
\tilde{h} \overset{\text{def}}{=} \frac{1}{g} \sum_{s \in G} \rho^2_s \circ h \circ (\rho^1_s)^{-1},
\]
c.f. Lemma 1.29. Then the following holds:

(i) If \( \rho^1 \) and \( \rho^2 \) are not isomorphic, we have \( \tilde{h} = 0 \).

(ii) If \( V \overset{\text{def}}{=} V_1 = V_2 \) and \( \rho \overset{\text{def}}{=} \rho^1 = \rho^2 \), then \( \tilde{h} = \frac{\text{tr}(h)}{\dim V} \text{id}_V \). In particular, the map \( \text{Hom}_{\mathbb{C}[G]}(V_1, V_2) \to \mathbb{C} \) defined by \( h \mapsto \frac{\text{tr}(h)}{\dim V} \text{id}_V \) is an isomorphism of vector spaces.

**Proof.** Since \( \tilde{h} \) is \( G \)-linear:
\[
\rho^2_s \circ \tilde{h} \circ (\rho^1_s)^{-1} = \frac{1}{g} \sum_{s \in G} \rho^2_s \circ \rho^2_s \circ h \circ (\rho^1_s)^{-1} \circ (\rho^1_s)^{-1} = \frac{1}{g} \sum_{s \in G} \rho^2_s \circ h \circ (\rho^1_s)^{-1} = \tilde{h},
\]
Example 1.35.  

We have

\[
\text{tr} (\bar{h}) = \frac{1}{g} \sum_{s \in G} \text{tr} (\rho_s \circ h \circ \rho_s^{-1}) = \frac{1}{g} \sum_{s \in G} \text{tr} (h) = \text{tr} (h),
\]

and since \( \text{tr} (\bar{h}) = \lambda \text{id}_V \), we conclude that \( \lambda = \frac{\text{tr} (\bar{h})}{\dim V} = \frac{\text{tr} (h)}{\dim V}. \) The last statement follows since for a \( G \)-linear map \( h : V_1 \to V_2 \), we have \( \bar{h} = h. \)

**Corollary 1.34.** (The matrix representation of Schur’s Lemma) Let \( V_1, V_2 \) be two irreducible representations of \( G \) and set \( g \defeq \text{ord}(G) \). Pick bases \( \{e_1, \ldots, e_m\} \) and \( \{f_1, \ldots, f_n\} \) of \( V_1 \) and \( V_2 \) respectively, so that \( R_1^1 = (r_{i,j_1}^1(s)) \) and \( R_2^2 = (r_{i,j_2}^2(s)). \)

(i) If \( V_1 \) and \( V_2 \) are not isomorphic representations, then for all \( 1 \leq i_1, j_1 \leq m, 1 \leq i_2, j_2 \leq n \), we have

\[
\frac{1}{g} \sum_{s \in G} r_{i,j_2}^2(s) r_{i_1,j_1}^1(s^{-1}) = 0.
\]

(ii) In the case where \( V \defeq V_1 = V_2 \) (so that \( m = n \)), for all \( 1 \leq i_1, j_1, i_2, j_2 \leq n \),

\[
\frac{1}{g} \sum_{s \in G} r_{i,j_2}^2(s) r_{i_1,j_1}^1(s^{-1}) = \frac{1}{\dim V} \delta_{i_1,j_2} \delta_{i_2,j_1}.
\]

**Proof.** Let \( h : V_1 \to V_2 \) be a linear map written as \( (x_{ij}) \) matrix form and similarly write \( \bar{h} \) as \( (x_{ij}^0) \).

Then

\[
x_{i_2,j_1}^0 = \frac{1}{g} \sum_{s \in G} \left( r_{i,j_2}^2(s) (x_{ij})(r_{i,j_1}^1(s^{-1})) \right) = \frac{1}{g} \sum_{s \in G} \left( \frac{1}{g} \sum_{s \in G} r_{i,j_2}^2(s) r_{i_1,j_1}^1(s^{-1}) \right) x_{i_2,j_1}.
\]

Picking the linear map \( h \) specified by \( x_{ij} = \delta_{ij} \delta_{i_2,j_1} \), since \( \bar{h} = 0 \) in part (i) of Schur’s lemma, we get that the double sum reduces to

\[
\frac{1}{g} \sum_{s \in G} r_{i,j_2}^2(s) r_{i_1,j_1}^1(s^{-1}) = x_{i_2,j_1}^0 = 0,
\]

which proves (i). Using the same map \( h \) in part (ii), we obtain \( x_{i_2,j_1}^0 = \frac{\text{tr}(h)}{\dim V} \delta_{i_2,j_1} = \frac{1}{\dim V} \delta_{i_1,j_2} \delta_{i_2,j_1} \) by Schur’s lemma, which completes the proof.

**Example 1.35.**

- Consider a group \( G \) and a representation \( \rho : G \to \text{GL}(V) \). For \( s \in G \), we get a linear map \( \rho_s : V \to V \). When is this map \( G \)-linear? In other words, when does the following hold:

\[
\forall t \in G, \quad \rho_t \circ \rho_s = \rho_s \circ \rho_t.
\]

If \( G \) is abelian, the above condition trivially holds since

\[
\rho_s \circ \rho_t = \rho_{st} = \rho_t \circ \rho_s.
\]

By Schur’s Lemma, for every irreducible representation \( \rho : G \to \text{GL}(V), \rho_s = \lambda_s \text{id}_V \) for some \( \lambda_s \in \mathbb{C} \) and each \( s \in G \). If \( W \subset V \) is a non-zero vector subspace, then \( \rho_s(W) \subset W \) (because
Complex representations of finite groups

\[ \rho_\lambda = \lambda \cdot \text{id}_V \], hence \( W \) is \( G \)-stable; it follows that \( W = V \), i.e. \( V \) is one-dimensional. We conclude that the irreducible representations of an abelian group \( G \) are elements of its **Pontryagin dual group**:

\[ \hat{G} \overset{\text{def}}{=} \{ \rho : G \to \mathbb{S}^1 \mid \rho \text{ is a group homomorphism} \} \]

where \( \mathbb{S}^1 \subseteq \mathbb{C}^\times \) is the subgroup of \( \mathbb{C}^\times \) of complex numbers \( z \) satisfying \( |z| = 1 \).

Consider the group \( G = S_3 \overset{\text{def}}{=} \langle \sigma, \tau \mid \sigma^3 = \tau^2 = 1, \sigma \tau = \tau \sigma^2 \rangle \). We know two representations of \( G \) of degree 1, namely the trivial representation and the alternating or sign representation, which is just the homomorphism \( \pi : S_3 \to \mathbb{C}^\times \) which maps a permutation to its sign, which lands in \( \{1, -1\} \).

There is another irreducible representation of \( S_3 \), which we compute as follows. The group \( S_3 \) acts on the vector space \( \mathbb{C}^3 \) (for which we give the basis \( \langle e_1, e_2, e_3 \rangle \) by \( \sigma \cdot e_i = e_{\sigma(i)} \). In coordinates, \( \sigma(z_1, z_2, z_3) = (z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}) \). To see this,

\[ x = \sum_{i=1}^3 z_i e_i \quad \implies \quad \sigma(x) = \sum_{i=1}^3 z_i \sigma(e_i) = \sum_{i=1}^3 z_i e_{\sigma(i)} = \sum_{i=1}^3 z_{\sigma^{-1}(i)} e_i \]

(c.f. Example 1.5). The subspace \( W = \langle (1,1,1) \rangle_\mathbb{C} \) is of course \( G \)-stable and of dimension one, so since the standard inner product of \( \mathbb{C}^3 \) is \( S_3 \)-invariant, the orthogonal complement \( V \overset{\text{def}}{=} \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0 \} \) is also \( G \)-stable by Remark 1.14 (or by direct computation), leaving us with \( \mathbb{C}^3 = V \oplus W \). The representation \( V \) is called the **standard representation** of \( S_3 \). It is clearly irreducible, since it has dimension 2 and if it were reducible, there would be a \( G \)-stable subspace of dimension 1; if \( x = \sum_{i=1}^3 z_i e_i \) is the generator, then

\[ \sum_{i=1}^3 z_i e_{\sigma(i)} = \sigma \cdot \sum_{i=1}^3 z_i e_i = \lambda_\sigma \sum_{i=1}^3 z_i e_i, \]

hence for \( \sigma = (ij) \) we have \( \lambda_\sigma = \pm 1 \) since \( \sigma^2 = \text{id} \). It follows that \( z_1 = \pm z_2 = \pm z_3 \) by comparing coefficients, and since \( z_1 + z_2 + z_3 = 0 \), we reach a contradiction since \( z_1 \pm z_1 \pm z_1 \in \{-3z_1, -z_1, z_1, 3z_1\} \) (c.f. Theorem 2.46 for a generalization).
Chapter 2

Character theory

2.1 Complex linear algebra review: the spectral theorem

Before we start developing character theory over $\mathbb{C}$, we make a quick review of complex linear algebra and the spectral theorem for normal matrices.

Definition 2.1. Let $n \geq 1$ be an integer and $A$ be an $n \times n$ complex matrix. The conjugate transpose (also called hermitian transpose) of $A = (a_{ij})$ is the matrix $A^\dagger$ defined by $A^\dagger_{ij} = \overline{a_{ji}}$. The matrix $A$ is said to be

(i) **hermitian** if $A = A^\dagger$
(ii) **normal** if $AA^\dagger = A^\dagger A$
(iv) **unitary** if it is normal and invertible, in which case $A^{-1} = A^\dagger$ since $AA^\dagger = A^\dagger A = I$; equivalently, $A$ has orthonormal rows (resp. columns) under the standard inner product in $\mathbb{C}^n$
(v) **diagonal** if there exists a basis $\{e_1, \cdots, e_n\}$ of $\mathbb{C}^n$ and complex numbers $\{\lambda_1, \cdots, \lambda_n\}$ such that $Ae_i = \lambda_i e_i$ for $i = 1, \cdots, n$
(vi) **diagonalizable** if there exists an $n \times n$ matrix $P$ such that $PAP^{-1}$ is diagonal
(vii) **unitarily diagonalizable** if there exists an $n \times n$ unitary matrix $U$ such that $UAU^\dagger$ is diagonalizable (note that $UAU^\dagger = UAU^{-1}$).
(viii) **upper triangular** (resp. **lower triangular**) if $a_{ij} = 0$ for $i > j$ (resp. $i < j$)
(ix) **triangular** when it is either upper or lower triangular.

Recall that $\text{End}_\mathbb{C}(V) \overset{\text{def}}{=} \text{Hom}_\mathbb{C}(V, V)$ is the set of linear endomorphisms of $V$. We can develop the analogous concepts for a linear endomorphism $f \in \text{End}_\mathbb{C}(V)$. The **adjoint** of $f$ is the linear endomorphism corresponding to the conjugate transpose of its matrix over a basis, and is defined as the unique linear map $f^\dagger$ such that for any $v, w \in V$,

$$(f(v) \mid w) = (v \mid f^\dagger(w)).$$

(This determines $f^\dagger$ uniquely since we can use the above equation to compute its coefficient matrix, which turns out to be $A^\dagger$ when $f$ has coefficient matrix $A$.) The endomorphism $f$ is

(i)" **hermitian** (also called **autoadjoint**) if $f = f^\dagger$, i.e. if for any $v, w \in V$, we have

$$(f(v) \mid w) = (v \mid f(w))$$
(ii)* **normal** if \( f \) and \( f^\dagger \) commute, namely \( f \circ f^\dagger = f^\dagger \circ f \)

(iv)* **unitary** if there exists a basis \( \{ v_1, \ldots, v_n \} \) of \( V \) for which the vectors \( \{ f(v_1), \ldots, f(v_n) \} \) form an orthonormal basis of \( V \), namely \( \langle f(v_i), f(v_j) \rangle = \delta_{ij} \)

(vii)* **unitarily diagonalizable** if there exists an orthonormal basis of eigenvectors for \( f \).

**Remark 2.2.** transpose matrix \( A \) diagonal hermitian matrix is the same as a real diagonal matrix. Diagonal, hermitian matrices and unitary matrices are all normal. However, there are matrices which are neither of those and are still normal: as an example, consider

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\]

which is normal since

\[
AA^\dagger = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{bmatrix} = A^\dagger A
\]

but it is neither hermitian, diagonal nor unitary. Also note that just as for real matrices, we had \((AB)^\dagger = B^\dagger A^\dagger\) where \(A^\dagger\) denotes the transpose of \( A \), for two complex matrices, we have \((AB)^\dagger = B^\dagger A^\dagger\).

**Lemma 2.3.** Let \( A \) be an \( n \times n \) complex matrix. If \( A \) is normal and triangular, then \( A \) is diagonal.

**Proof.** Without loss of generality, assume \( A \) is upper triangular. The vector \( Ae_n \) is the \( n \)th column of \( A \) and \( A^\dagger e_n \) is the \( n \)th row vector of \( A \) on which conjugation has been applied on the coefficients. We have

\[
\|Ae_n\|^2 = (Ae_n | Ae_n) = (e_n | A^\dagger Ae_n) = (e_n | AA^\dagger e_n) = (A^\dagger e_n, A^\dagger e_n) = \|A^\dagger e_n\|^2.
\]

Since the coefficient in front of \( e_n \) is the only non-zero coefficient of \( A^\dagger e_n \), the coefficient in front of \( e_n \) of \( Ae_n \) also has to be its only non-zero coefficient (because their \( n \)th coefficients are conjugates, so the contrary would imply \( \|Ae_n\|^2 > \|A\|^2 \)). It follows that \( Ae_n = \lambda_n e_n \) and \( Ae_i \in \langle e_1, \ldots, e_{n-1} \rangle \) for \( i = 1, \ldots, n-1 \). The \((n-1) \times (n-1)\) block submatrix of \( A \) is also normal (since \( A \) was), so we conclude by induction on \( n \) since the case \( n = 1 \) is trivial.

**Proposition 2.4.** Let \( A \) be an \( n \times n \) complex matrix. There exists an upper triangular (resp. lower triangular) matrix \( T \) and a unitary matrix \( U \) such that \( A = UTU^\dagger \). A pair \((U, T)\) for which this holds is called an upper Schur decomposition for \( A \) (resp. lower Schur decomposition ; the name “Schur decomposition” usually stands for an upper Schur decomposition).

**Proof.** Without loss of generality, we prove the result for upper Schur decompositions ; otherwise, one can apply the result to \( A^\dagger \) instead and apply \((-)^\dagger \) on the equation \( A^\dagger = UT^\dagger U^\dagger \) to recover the lower triangular case.

The goal is to find an orthonormal basis in which the expression of \( A \) is upper triangular (orthonormality guarantees that the base change matrix will be unitary). Let \( \lambda \) be an eigenvalue for \( A \) and \( v \) a corresponding eigenvector. Set \( v_1 \stackrel{def}{=} v \) and let \( V_2 \) be the orthogonal subspace to \( \langle v_1 \rangle \) in \( \mathbb{C}^n \) of dimension \( n-1 \). After choosing some orthonormal basis of \( V_2 \) and adding \( v_1 \) to it to obtain an orthonormal basis of \( \mathbb{C}^n \), the matrix \( A \) has the following form over this orthonormal basis:

\[
\begin{pmatrix}
\lambda & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]
so applying the result by induction on \( n \) on the block matrix \( B \) of rows and columns \( 2,3,\cdots,n \), we obtain an orthonormal basis \( \{ v_2,\cdots,v_n \} \) of \( V_2 \) such that \( B \) is upper triangular when expressed over that basis. It follows that \( A \) is upper triangular over this basis. The condition on \( v \) states that \( A \) is unitary, hence unitarily diagonalizable by the spectral theorem. The corresponding basis over which \( \rho \) stabilizes, namely \( f(V_i) \subseteq V_i \). The proof is equally easy and is left to the interested reader; use the above proof as a sketch.

Remark 2.5. The proof of Proposition 2.4 could have been equally well done in the language of complete flags. A complete flag in \( V \) is a filtration \( (V_0,V_1,\cdots,V_n=V) \) of subspaces of \( V \), namely \( V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n= V \) where \( \dim V_i = i \) for \( i = 0,\cdots,n \) (when we remove the dimension condition but only ask for strict inclusions, we speak of a flag). In this language, Proposition 2.4 is equivalent to the fact that if \( f \) is a linear endomorphism of \( V \), there exists a complete flag \( (V_0,\cdots,V_n) \) which \( f \) stabilizes, namely \( f(V_i) \subseteq V_i \). The proof is equally easy and is left to the interested reader; use the above proof as a sketch.

Notice that no notion of orthogonality was involved in the proof using complete flags; to recover the orthonormal basis, just take \( v_1 \in V_1 \setminus \{0\} \) and build an orthonormal basis of \( V \) by picking \( v_i \in V_i \setminus V_{i-1} \) orthogonal to the previously chosen basis vectors \( v_1,\cdots,v_{i-1} \) (this can be done via the Gram-Schmidt algorithm). The fact that \( f \) stabilizes the flag ensures that the corresponding matrix is upper triangular.

Theorem 2.6. (Spectral theorem) Let \( A \) be an \( n \times n \) complex matrix. Then \( A \) is normal if and only if it is unitarily diagonalizable, i.e. if and only if there exists a Schur decomposition \( (U,T) \) where \( T \) is diagonal.

\textbf{Proof.} (\( \Leftarrow \)) If \( A = UDU^\dagger \) where \( U \) is unitary and \( D \) is diagonal, since \( D \) is normal, so is \( A \):

\[ AA^* = (UDU^\dagger)(UDU^\dagger)^\dagger = UDU^\dagger UD^\dagger U^\dagger = UDD^\dagger U^\dagger = UU^\dagger = \cdots = A^*A. \]

(\( \Rightarrow \)) Let \( (U,T) \) be a Schur decomposition. Since \( A = UTU^\dagger \), we have \( UAU^\dagger = T \). By a computation similar as the one proving the direction (\( \Leftarrow \)), one shows that \( T \) is normal; since it is upper triangular, it is diagonal by Lemma 2.3.

Corollary 2.7. Let \( V \) be a complex inner product space and \( f : V \to V \) a linear endomorphism such that for any \( v,w \in V \), we have \( (f(v) \mid f(w)) = (v \mid w) \). There exists an orthonormal basis for \( V \) over which \( f \) is diagonal.

\textbf{Proof.} Let \( \{v_1,\cdots,v_n\} \) be an arbitrary orthonormal basis for \( V \) and consider the matrix form \( A \) of \( f \) over this basis. The condition on \( f \) states that \( A \) is unitary since its columns are orthonormal (because \( \{v_1,\cdots,v_n\} \) was chosen orthonormal). Therefore \( A \) is normal, hence unitarily diagonalizable by the spectral theorem. The corresponding basis over which \( f \) is diagonal has to be orthonormal since the base change matrix we found was unitary.

We can now turn our attention back to complex linear representations. Note that Corollary 2.7 states that when \( \rho : G \to \text{GL}(V) \) is a representation and \( (-\mid-) \) is a \( G \)-invariant inner product on \( V \), then \( \rho \) maps each element of \( G \) to a unitary diagonalizable automorphism of \( V \), which is why complex linear representations are sometimes called unitary representations.

2.2 Row orthogonality relations

The goal of section is to develop the elementary properties of characters and show that the character of a representation completely determines a representation and helps understand Theorem 1.27, i.e. the decompositions of a representation in a direct sum of irreducible subrepresentations. We will prove that if \( V = \bigoplus_{i=1}^m W_i \) is such a decomposition, given an irreducible representation \( W \) of \( G \), the number of \( 1 \leq i \leq m \) such that \( W_i \cong W \) does not depend on the chosen representation, giving a sense of uniqueness.
of this decomposition. The main tool to achieve this, as the title of the section says, will be the row orthogonality relations (c.f. Theorem 2.19).

**Definition 2.8.** Given a complex linear representation \( \rho : G \rightarrow \text{GL}(V) \), define the **character** of \( \rho \) by

\[
\chi : G \rightarrow \mathbb{C}, \quad s \mapsto \chi(s) \overset{\text{def}}{=} \text{tr}(\rho_s).
\]

A character is called **irreducible** if it is the character of an irreducible representation. Its **degree** is defined as \( \deg \chi \overset{\text{def}}{=} \chi(1) \). (It corresponds to the degree of the representation we used to define it.)

**Proposition 2.9.** If \( \chi \) is the character of the representation \( \rho : G \rightarrow \text{GL}(V) \), then

(i) \( \chi(1) = \dim V = \deg \rho \).

(ii) For all \( s \in G \), we have \( \chi(s^{-1}) = \overline{\chi(s)} \).

(iii) For all \( s, t \in G \), we have \( \chi(tst^{-1}) = \chi(s) \).

**Proof.**

(i) We have \( \chi(1) = \text{tr}(\rho_1) = \text{tr}(\text{id}_V) = \dim V = \deg \rho \).

(ii) Since \( s \in G \) has finite order, \( \rho_s^n = \text{id}_V \), hence if \( \lambda \) is an eigenvalue of \( \rho_s \), \( \lambda^n = 1 \), e.g. \( |\lambda| = 1 \), which implies \( \lambda^{-1} = \overline{\lambda} \). The eigenvalues of \( \rho_{s^{-1}} \) are just \( \lambda^{-1} \) for \( \lambda \) an eigenvalue of \( \rho_s \) (because \( \rho_s(v) = \lambda v \iff \lambda^{-1} v = \rho_{s^{-1}}(v) \)). It follows that

\[
\overline{\chi(s)} = \text{tr}(\rho_s) = \sum_{\lambda} \lambda^{-1} = \text{tr}(\rho_{s^{-1}}) = \chi(s^{-1}).
\]

(iii) This follows from properties of the trace map explained in Definition 1.31.

**Proposition 2.10.** Let \( \rho^i : G \rightarrow \text{GL}(V^i) \) and \( \chi_i \) be its character for \( i = 1, 2 \). Then

(i) The character \( \chi \) of \( V \overset{\text{def}}{=} V_1 \oplus V_2 \) is \( \chi_1 + \chi_2 \).

(ii) The character \( \psi \) of \( V \overset{\text{def}}{=} V_1 \otimes V_2 \) is \( \chi_1 \otimes \chi_2 = \chi_1 \chi_2 \), where the latter is given by pointwise multiplication \( \chi_1 \chi_2(s) \overset{\text{def}}{=} \chi_1(s) \chi_2(s) \).

**Proof.**

(i) \( \rho^1 \) and \( \rho^2 \) are given in matrix form by matrices \( R^1_s \) and \( R^2_s \) over some bases for \( V_1 \) and \( V_2 \), and in block form, the matrix \( R_s \) of the representation \( \rho = \rho^1 \oplus \rho^2 \) is given by

\[
R_s = \begin{bmatrix} R^1_s & 0 \\ 0 & R^2_s \end{bmatrix} \quad \Rightarrow \quad \text{tr}(R_s) = \text{tr}(R^1_s) + \text{tr}(R^2_s),
\]

from which we deduce the result.
Let \( \rho = \rho^1 \otimes \rho^2 \), \( R^i_s \) is the matrix for \( \rho^i \) and \( R_s \) is the matrix for \( \rho \). By Remark 1.19,

\[
\chi(s) = \text{tr} \left( R_s \right) = \sum_{i_1=1}^{n} \sum_{i_2=1}^{m} r^1_{i_1 i_1}(s) r^2_{i_2 i_2}(s)
= \left( \sum_{i_1=1}^{n} r^1_{i_1 i_1}(s) \right) \left( \sum_{i_2=1}^{m} r^2_{i_2 i_2}(s) \right)
= \text{tr} \left( R^1_s \right) \text{tr} \left( R^2_s \right)
= \chi_1(s) \chi_2(s).
\]

**Theorem 2.11.** Let \( \rho : G \to \text{GL}(V) \) be a representation of the finite group \( G \) and set \( g \overset{\text{def}}{=} \text{ord}(G) \), \( n \overset{\text{def}}{=} \dim V \). The endomorphisms \( \rho_s \) are unitarily diagonalizable. The character \( \chi(s) \) is a sum of \( n \) roots of unity of order \( g \), i.e. complex numbers \( \omega \) satisfying \( \omega^g = 1 \). In particular, \( |\chi(s)| \leq n \) and \( \chi(s) = n \) holds if and only if \( \rho_s = \text{id}_V \).

**Proof.** By Remark 1.14, we can choose a \( G \)-invariant inner product on \( V \) and a basis \( \{ e_1, \ldots, e_n \} \) which is orthonormal with respect to \( (- | -) \). The columns of \( \rho_s \) in its matrix form over this basis are \( \rho_s(e_1), \ldots, \rho_s(e_n) \), and they are orthogonal under \( (- | -) \) by definition since this inner product has been taken \( G \)-invariant and our basis is orthonormal. By Corollary 2.7, \( \rho_s \) is unitarily diagonalizable. Since \( s \in G \) has finite order, then \( \rho_s^{\text{ord}(s)} = \text{id}_V \). The eigenvalues of \( \rho_s \) are therefore \( \text{ord}(s) \)th roots of unity, and summing them up shows that \( \chi(s) \) is a sum of \( n \) roots of unity. Since \( \text{ord}(s) \) divides \( g \) by Lagrange’s theorem, the eigenvalues of \( \rho_s \) are \( g \)th roots of unity.

Write \( \chi(s) = \omega_1 + \cdots + \omega_n \) where \( \omega_i \) is a \( g \)th root of unity. Then by the triangle inequality,

\[
|\chi(s)| = |\omega_1 + \cdots + \omega_n| \leq |\omega_1| + \cdots + |\omega_n| = 1 + \cdots + 1 = n.
\]

The statement \( \chi(s) = n \) is equivalent to \( \omega_i = 1 \) for \( 1 \leq i \leq n \) (because \( \chi(s) = n \) implies \( \text{Re}(\chi(s)) = n \)), which means \( \rho_s = \text{id}_V \).

**Proposition 2.12.** Let \( \rho : G \to \text{GL}(V) \) be a representation and \( \chi \) its character. Write \( \chi^2_\sigma \) for the character of \( \text{Sym}^2(V) \) and \( \chi^2_\alpha \) be the character of \( \text{Alt}^2(V) \) (c.f. Example 1.21). Then

\[
\chi^2_\sigma(s) = \frac{1}{2} \left( \chi(s)^2 + \chi(s^2) \right), \quad \chi^2_\alpha(s) = \frac{1}{2} \left( \chi(s)^2 - \chi(s^2) \right).
\]

**Proof.** Let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues of \( \rho_s \), so that the eigenvalues of \( \rho_s^2 = \rho^2_s \) are \( \lambda^2_1, \ldots, \lambda^2_n \). Choose a basis \( \{ e_1, \ldots, e_n \} \) of \( V \) which consists of eigenvectors for \( \rho_s \); this is always possible by Theorem 2.11. Recall Example 1.21 where we have shown, using the linear endomorphism \( \theta \) of \( V \otimes V \) given by \( \theta(v \otimes w) = w \otimes v \), that \( V \otimes V = \text{Sym}^2(V) \oplus \text{Alt}^2(V) \). The sets

\[
\{ e_i \otimes e_j + e_j \otimes e_i \mid 1 \leq i \leq j \leq n \} \subseteq \text{Sym}^2(V), \quad \{ e_i \otimes e_j - e_j \otimes e_i \mid 1 \leq i < j \leq n \} \subseteq \text{Alt}^2(V)
\]

are obviously linearly independent, and since \( \dim V = n^2 = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} \), we see that they generate their respective subspaces in which we wrote containment. It follows that they form bases of \( \text{Sym}^2(V) \) and \( \text{Alt}^2(V) \), respectively. Now

\[
(\rho_s \otimes \rho_s)(e_i \otimes e_j) = \rho_s(e_i) \otimes \rho_s(e_j) = (\lambda_i e_i) \otimes (\lambda_j e_j) = \lambda_i \lambda_j (e_i \otimes e_j),
\]

so that \( \rho_s \otimes \rho_s \) is a diagonal matrix over the bases given for \( \text{Sym}^2(V) \) and \( \text{Alt}^2(V) \); in both cases, the
We endow $V$ with the trivial representation $1$, that is, the representation $1: G \to GL(V)$ defined by $1_g(v) = v$ for all $g \in G$.

In particular, when $V$ is a representation of the finite group $G$, we define the character of $V$ to be the function $\chi(\rho): G \to \mathbb{C}$ given by $\chi(\rho)(g) = \chi(V)(g)$, where $\chi(V)$ is the character of $V$.

The character of a representation $V$ is a function on the group $G$, and it encodes important information about the representation. The character of the trivial representation of $G$ is a constant function equal to the order of $G$.

Theorem 2.13. Let $\rho: G \to GL(V)$ be a representation of a finite group $G$. Set

$$V^G \overset{def}{=} \{ v \in V \mid \forall t \in G, \; \rho_t(v) = v \}.$$ 

Set $g = \text{ord}(G)$ and consider

$$f \overset{def}{=} \frac{1}{g} \sum_{t \in G} \rho_t \in \text{Hom}_\mathbb{C}(V, V).$$

The map $f: V \to V$ is a $G$-linear projection of $V$ onto $V^G$, meaning that $V^G$ is a subrepresentation of $V$ equal to the sum of all trivial subrepresentations of $V$.

**Proof.** By definition, if $W \leq V$ is a trivial subrepresentation, then $W \leq V^G$. We know that $f$ is $G$-linear since

$$\rho_t \circ f = \rho_t \circ \left( \frac{1}{g} \sum_{s \in G} \rho_s \right) = \frac{1}{g} \sum_{s \in G} \rho_t \circ \rho_s = \frac{1}{g} \sum_{s \in G} \rho_{st} = \frac{1}{g} \sum_{s \in G} \rho_s \circ \rho_t = f \circ \rho_t.$$

It follows that $\text{im } f \subseteq V^G$. Conversely, if $v \in V^G$, $\rho_s(v) = v$ for all $s \in G$, hence $f(v) = v$; therefore $\text{im } f = V^G$. Since $f(v) \in V^G$, $f^2 = f$, so $f$ is a projection onto $V^G$.

Corollary 2.14. Let $V$ be a representation of the finite group $G$ and set $g = \text{ord}(G)$. Under a decomposition of $V$ as a direct sum of irreducible representations, the number of copies of the trivial representation in $V$ in this decomposition equals $\dim V^G$. Let $\chi$ be the character of $V$. Then

$$\dim V^G = \text{tr } (f) = \frac{1}{g} \sum_{t \in G} \text{tr } (\rho_t) = \frac{1}{g} \sum_{t \in G} \chi(t).$$

Definition 2.15. Let $\rho^V: G \to GL(V)$ and $\rho^W: G \to GL(W)$ be two representations of the finite group $G$. We turn the complex vector space $\text{Hom}_\mathbb{C}(V, W)$ into a representation $\rho: G \to GL(\text{Hom}_\mathbb{C}(V, W))$ as follows: given $f: V \to W$ and $s \in G$,

$$\rho_s(f) \overset{def}{=} \rho^W_s \circ f \circ (\rho^V_s)^{-1}.$$ 

In particular, when $\rho: G \to GL(V)$ is a representation of $G$ and $W = \mathbb{C}$ is the trivial representation, we endow $V^* \overset{def}{=} \text{Hom}_\mathbb{C}(V, \mathbb{C})$ with this representation $\rho^* : G \to GL(V^*)$, so that for $f \in V^*$, $\rho_s^*(f) \overset{def}{=} f \circ (\rho^V_s)^{-1}$; we call it the dual representation.

Proposition 2.16. Let $V, W$ be two representations of the group $G$.

\[ 2 \left( \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \right) - \left( \sum_{i=1}^{n} \lambda_i^2 \right) = 2 \left( \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \right) + \left( \sum_{i=1}^{n} \lambda_i^2 \right), \]

we have

$$\chi_{\alpha}^2(s) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \frac{1}{2} \left( \left( \sum_{i=1}^{n} \lambda_i \right)^2 + \left( \sum_{i=1}^{n} \lambda_i^2 \right) \right) = \frac{\chi(s)^2 + \chi(s^2)}{2} \quad \text{and}$$

$$\chi_{\alpha}^2(s) = \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = \frac{1}{2} \left( \left( \sum_{i=1}^{n} \lambda_i \right)^2 - \left( \sum_{i=1}^{n} \lambda_i^2 \right) \right) = \frac{\chi(s)^2 - \chi(s^2)}{2}.$$
Proposition 2.9. \( \text{The point of this} \)
\( \text{Remark 2.17.} \)
\( \text{Chapter 2} \)

(i) The natural isomorphism \( \Phi : V^* \otimes W \cong \text{Hom}_\mathbb{C}(V, W) \) of Definition 1.31 is an isomorphism of representations.

(ii) If \( \chi_V \) is the character of \( V \), then \( \chi_{V^*} = \overline{\chi_V} \).

(iii) If \( \chi_{\text{Hom}} \) is the character of \( \text{Hom}_\mathbb{C}(V, W) \), then \( \chi_{\text{Hom}} = \overline{\chi_V \chi_W} \).

**Proof.**

(i) Let \( s \in G, v \in V, \varphi \in V^* \) and \( w \in W \). Then

\[
\Phi((s \cdot \varphi \otimes (s \cdot w))(v) = \Phi((s \cdot \varphi) \otimes (s \cdot w))(v) = s \cdot \Phi((s \cdot \varphi \otimes (s \cdot w))(v)) = (s \cdot \Phi(\varphi \otimes w))(v).
\]

(ii) If \( \{v_1, \ldots, v_n\} \) is an eigenbasis for \( \rho_s^V \), then the dual basis \( \{\omega_1, \ldots, \omega_n\} \) is an eigenbasis for \( \rho_s^{V^*} \) and if \( v_j \) is an eigenvector of \( \rho_s^V \) with eigenvalue \( \lambda_i \), then \( \omega_j \) is an eigenvector of \( \rho_s^{V^*} \) with eigenvalue \( \lambda_i^{-1} \):

\[
(s \cdot \omega_i)(v_j) = \omega_i(s^{-1} \cdot v_j) = \omega_i(\lambda_j^{-1} v_j) = \lambda_j^{-1} \delta_{ij} = \lambda_i^{-1} \delta_{ij} \quad \implies \quad s \cdot \omega_i = \lambda_i^{-1} \omega_i.
\]

Summing over all algebraic eigenvalues of \( \rho_s^V \), we obtain \( \chi_{V^*} = \overline{\chi_V} \).

(iii) Since \( \text{Hom}_\mathbb{C}(V, W) \) and \( V^* \otimes W \) are isomorphic representations, they have the same character. By Proposition 2.10 (ii) and part (ii) of this proposition,

\[
\chi_{\text{Hom}} = \chi_{V^* \otimes W} = \chi_{V^*} \chi_W = \overline{\chi_V \chi_W}.
\]

**Remark 2.17.** The point of this \( G \)-action on \( \text{Hom}_\mathbb{C}(V, W) \) is that

\[
\text{Hom}_\mathbb{C}(V, W)^G = \{ f : V \to W \mid \forall s \in G, \quad \rho_s^W \circ f \circ (\rho_s^V)^{-1} = f \} = \{ f : V \to W \mid \forall s \in G, \quad \rho_s^W \circ f = f \circ \rho_s^V \} = \text{Hom}_{\mathbb{C}[G]}(V, W).
\]

Note that the \( G \)-linear projection \( \rho : \text{Hom}_\mathbb{C}(V, W) \to \text{Hom}_\mathbb{C}(V, W) \) onto \( \text{Hom}_{\mathbb{C}[G]}(V, W) = \text{Hom}_\mathbb{C}(V, W)^G \) of Theorem 2.13 is exactly the same map we used in the proofs of Maschke’s theorem and Corollary 1.33. Combining the above equality with Theorem 2.13, we obtain

\[
\dim \text{Hom}_{\mathbb{C}[G]}(V, W) = \dim \text{Hom}_\mathbb{C}(V, W)^G = \frac{1}{g} \sum_{s \in G} \chi_{\text{Hom}}(s) = \frac{1}{g} \sum_{s \in G} \overline{\chi_V(s) \chi_W(s)}.
\]

If \( V, W \) are irreducible, then by Schur’s lemma,

\[
\dim \text{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \text{ in } \text{Rep}_\mathbb{C}(G) \\ 0 & \text{otherwise.} \end{cases}
\]

**Definition 2.18.** Consider the complex vector space

\[
\mathbb{C}_{cl}[G] \overset{\text{def}}{=} \{ f : G \to \mathbb{C} \mid \forall s, t \in G, \quad f(sts^{-1}) = f(t) \},
\]

called the \textbf{vector space of class functions} on \( G \). If \( \overline{g} = \text{ord}(G) \) and \( \alpha, \beta \) are class functions of \( G \), then

\[
(\alpha, \beta) \overset{\text{def}}{=} \frac{1}{\overline{g}} \sum_{s \in G} \alpha(s) \overline{\beta(s)},
\]

where \( \overline{\beta(s)} \overset{\text{def}}{=} \overline{\beta(s)} \), turns \( \mathbb{C}_{cl}[G] \) into an inner product space. Note that characters are class functions by Proposition 2.9.
Theorem 2.19. (Row orthogonality relations) Let $V, W$ be representations of $G$.

(i) In light of Remark 2.17, if $V$ and $W$ are irreducible, we have

$$
(\chi_V, \chi_W) = \frac{1}{g} \sum_{t \in G} \chi_V(t) \overline{\chi_W(t)} = \dim \text{Hom}_{\mathbb{C}[G]}(V, W) = \begin{cases} 
1 & \text{if } V \simeq W \\
0 & \text{otherwise.}
\end{cases}
$$

This shows that irreducible representations are determined up to $G$-linear isomorphism by their character; furthermore, the irreducible characters of $G$ are pairwise orthogonal in $\mathbb{C}_{cl}[G]$, which proves that a finite group $G$ admits only finitely many irreducible characters.

(ii) Write $V = \bigoplus_{i=1}^{h} W_i$ where the $W_i$ are irreducible subrepresentations of $V$. If $W$ is irreducible, then

$$
(\chi_V, \chi_W) = \dim \text{Hom}_{\mathbb{C}[G]}(V, W) = \#\{1 \leq i \leq h \mid W_i \simeq W \text{ in } \text{Rep}_{\mathbb{C}}(G)\}.
$$

In particular, $\#\{1 \leq i \leq h \mid W_i \simeq W \text{ in } \text{Rep}_{\mathbb{C}}(G)\}$ is independent of the choice of decomposition of $V$ as a direct sum of irreducible representations.

(iii) The number $(\chi_V, \chi_W)$ is a non-negative integer, and so in particular $(\chi_V, \chi_W) = \overline{(\chi_W, \chi_V)} = (\chi_W, \chi_V)$. This implies that the roles of $V$ and $W$ in (ii) can be reversed, namely, if $V$ is irreducible and $W = \bigoplus_{i=1}^{h} W'_i$ is a decomposition of $W$ into irreducible subrepresentations, then

$$
(\chi_V, \chi_W) = \dim \text{Hom}_{\mathbb{C}[G]}(V, W) = \#\{1 \leq i \leq h \mid W'_i \simeq V \text{ in } \text{Rep}_{\mathbb{C}}(G)\}.
$$

Proof. (i) The last equality follows from Remark 2.17. By the same remark, since dimensions are integers, we have

$$
\dim \text{Hom}_{\mathbb{C}[G]}(V, W) = \frac{\dim \text{Hom}_{\mathbb{C}}(V, W)^G}{g} = \frac{1}{g} \sum_{s \in G} \overline{\chi_V(s)} \chi_W(s) = \frac{1}{g} \sum_{s \in G} \chi_V(s) \overline{\chi_W(s)} = (\chi_V, \chi_W).
$$

The number of irreducible characters is thus finite since $\mathbb{C}_{cl}[G]$ is finite-dimensional.

(ii) This follows from part (i) and the fact that $\chi_W = \sum_{i=1}^{h} \chi_W$. The inner product $(\chi_V, \chi_W)$ does not require the direct sum decomposition to be defined, so that the number of irreducible subrepresentations isomorphic to $W$ appearing in a decomposition is also independent of the choice of such a decomposition.

(iii) Remark 2.17 implies that $(\chi_V, \chi_W) = \overline{\dim \text{Hom}_{\mathbb{C}[G]}(V, W)}$, so that $(\chi_V, \chi_W)$ is (the complex conjugate of) an integer, i.e. an integer. This integer has to be non-negative since by writing $V$ and $W$ as a direct sum of irreducible representations, $\chi_V$ and $\chi_W$ are a sum of irreducible characters, so by part (i), it equals a sum of 1’s and 0’s. The rest of the statement follows from part (ii).

Corollary 2.20. Let $V$ be a representation of $G$. Then $V$ is irreducible if and only if $(\chi_V, \chi_V) = 1$. In particular, $\chi_V$ is irreducible if and only if $\overline{\chi_V}$ is irreducible.
Chapter 2

**Proof.** Let $V = W_1 \oplus \cdots \oplus W_h$ be a decomposition of $V$ into irreducible representations. Then

$$
(\chi_V, \chi_V) = \sum_{i,j=1}^{h} (\chi_{W_i}, \chi_{W_j}) = \sum_{i,j=1}^{h} \delta_{ij} = h.
$$

It follows that $V$ is irreducible if and only if $(\chi_V, \chi_V) = h = 1$. For the last result, we note that

$$
(\overline{\chi}_V, \overline{\chi}_V) = \frac{1}{g} \sum_{s \in G} \overline{\chi}_V(s) \overline{\chi}_V(s) = \frac{1}{g} \sum_{s \in G} \overline{\chi}_V(s) \overline{\chi}_V(s) = (\chi_V, \chi_V).
$$

**Corollary 2.21.** If two representations $V_1, V_2$ of $G$ have the same character, they are isomorphic. A character thus determines a unique isomorphism class of representations of $G$.

**Proof.** Write $V_j = \bigoplus_{i=1}^{h} W_{i,j}$ for $j = 1, 2$ where the $W_{i,j}$ are irreducible subrepresentations of $V_j$. Given an irreducible representation $W$, since $(\chi_{V}, \chi_{W}) = (\chi_{V_2}, \chi_{W})$, the number of $W_{i,1}$ isomorphic to $W$ and the number of $W_{i,2}$ isomorphic to $W$ are equal. Considering all irreducible subrepresentations of $V_1$ and $V_2$, we see that $V_1$ and $V_2$ are isomorphic by summing the isomorphisms $W_{i,1} \simeq W_{\sigma(i),2}$ for some permutation $\sigma : \{1, \cdots, h\} \rightarrow \{1, \cdots, h\}$.

**Corollary 2.22.** Let $\chi_1, \cdots, \chi_h$ be the distinct irreducible characters corresponding to the isomorphism classes of the irreducible representations $W_1, \cdots, W_h$ in $\text{Rep}_C(G)$. If $V$ is a representation of $G$, then

$$
\chi_V = \sum_{i=1}^{h} (\chi_V, \chi_i) \chi_i.
$$

**Proof.** If we write $V = \bigoplus_{j=1}^{\ell} W'_j$ where the $W'_j$ are irreducible, then $\chi_{W'_j} = \chi_{i(j)}$ for a unique integer $i(j) \in \{1, \cdots, h\}$, which allows us to re-write $V \simeq \bigoplus_{j=1}^{\ell} W'_{i(j)}$. Therefore $\chi_V = \sum_{j=1}^{\ell} \chi_{i(j)}$, and gathering the characters of the sum which are equal into a single summand gives the result by Theorem 2.19 (ii) since there are precisely $(\chi_V, \chi_i)$ values of $j$ with $1 \leq j \leq \ell$ satisfying $i(j) = i$.

**Convention 2.23.** Until the end of this chapter, we are given a finite group $G$ with $g \overset{\text{def}}{=} \text{ord}(G)$ and we denote by $W_1, \cdots, W_h$ the distinct irreducible representations of $G$ with character $\chi_i$ of degree $n_i = \chi_i(1) = \dim W_i$. The conjugacy classes of $G$ are denoted by $C_1, \cdots, C_k$ and if $s \in C_i$, we write $\overline{C_s} \overset{\text{def}}{=} |C_i|$.  

**Definition 2.24.** Recall (c.f. Example 1.5) that $\mathbb{C}[G]$ is the regular representation of $G$ with basis $\{e_t\}_{t \in G}$ and $\rho_s(e_t) \overset{\text{def}}{=} e_{st}$. As a set, $\mathbb{C}[G] = \text{Hom}_{\mathbb{C}}(G, \mathbb{C})$ is the set of complex functions on $G$, hence we can equip $\mathbb{C}[G]$ with an inner product: for $f_1, f_2 \in \mathbb{C}[G]$, let

$$
(f_1, f_2) \overset{\text{def}}{=} \frac{1}{g} \sum_{s \in G} f_1(s) \overline{f_2(s)}.
$$

Defined this way, we see that $\mathbb{C}_s[G] \subseteq \mathbb{C}[G]$ is a subspace whose inner product is induced from that of $\mathbb{C}[G]$. Clearly, this inner product is $G$-invariant:

$$
(s \cdot f_1, s \cdot f_2) = \frac{1}{g} \sum_{t \in G} (s \cdot f_1)(t) \overline{(s \cdot f_2)(t)} = \frac{1}{g} \sum_{t \in G} f_1(s^{-1}t) f_2(s^{-1}t) = \frac{1}{g} \sum_{t \in G} f_1(t) f_2(t) = (f_1, f_2).
$$

**Proposition 2.25.** For $s, t \in G$, we have $(e_s, e_t) = 0$, hence $\{e_t\}_{t \in G}$ is an orthogonal basis of $\mathbb{C}[G]$.  


can normalize it, i.e. \( \{ \sqrt{g}e_t \}_{t \in G} \) is an orthonormal basis of \( \mathbb{C}[G] \). Furthermore,

\[
\chi_{\mathbb{C}[G]}(s) = \begin{cases} 
g & \text{if } s = 1 \\
0 & \text{otherwise.} \end{cases}
\]

**Proof.** For orthogonality of the basis, it suffices to notice that when \( s \neq t \), for any \( u \in G \), \( e_s(u)e_t(u) = 0 \) since \( e_s(u) \) or \( e_t(u) \) equals zero. When \( s = t \), we have

\[
(\sqrt{g}e_s, \sqrt{g}e_s) = \frac{\sqrt{g}^2}{g} \sum_{t \in G} e_s(t)e_s(t) = 1.
\]

We already know that \( \chi_{\mathbb{C}[G]}(1) = \dim \mathbb{C}[G] = g \) by Proposition 2.9. If \( s \in G \setminus \{ 1 \} \), then for all \( t \in G \), \( st \neq t \), thus \( \rho_s(e_t), e_t = (e_{st}, e_t) = 0 \), which means \( \chi_{\mathbb{C}[G]}(s) = \text{tr} \left( \rho_s \right) = 0 \).

**Theorem 2.26.** Every irreducible representation \( W_i \) is contained in \( \mathbb{C}[G] \) with multiplicity \( n_i = \dim W_i = \chi_i(1) \). In particular, \( \chi_{\mathbb{C}[G]} = \sum_{i=1}^{h} n_i \chi_i \).

**Proof.** By Proposition 2.25, we have \( (\chi_{\mathbb{C}[G]}, \chi_i) = \frac{1}{g} \sum_{s \in G} \chi_{\mathbb{C}[G]}(s) \overline{\chi_i(s)} = \overline{\chi_i(1)} = n_i \). Use Corollary 2.22.

**Corollary 2.27.** The degrees \( n_i \) of the irreducible representations of \( G \) satisfies \( g = \sum_{i=1}^{h} n_i^2 \). If \( s \in G \setminus \{ 1 \} \), then \( \sum_{i=1}^{h} n_i \chi_i(s) = \chi_{\mathbb{C}[G]}(s) = 0 \).

**Proof.** Substituting \( s = 1 \) in Theorem 2.26 gives

\[
g = \dim \mathbb{C}[G] = \chi_{\mathbb{C}[G]}(1) = \sum_{i=1}^{h} n_i \chi_i(1) = \sum_{i=1}^{h} n_i^2.
\]

By Proposition 2.25, substituting \( s \in G \setminus \{ 1 \} \) in Theorem 2.26 gives \( \sum_{i=1}^{h} n_i \chi_i(s) = \chi_{\mathbb{C}[G]}(s) = 0 \).

### 2.3 Column orthogonality relations and the canonical decomposition

**Lemma 2.28.** Let \( f \) be a class function on \( G \) and \( \rho : G \to \text{GL}(V) \) a representation with character \( \chi \). Define \( P_{\chi} : \mathbb{C}_{\text{cl}}[G] \to \text{End}_{\mathbb{C}}(V) \) by

\[
P_{\chi}(f) \overset{\text{def}}{=} \frac{1}{g} \sum_{t \in G} f(t)\rho_t^{-1}.
\]

If \( \rho \) is irreducible of degree \( n \), then \( P_{\chi}(f) = (\frac{1}{n})\text{id}_V \). In particular, \( \text{tr} \left( P_{\chi}(f) \right) = (f, \chi) \). Note : I chose the letter \( P_{\chi}(f) \) for “projection onto \( \chi \)” (because after taking the trace, this is the coefficient of the orthogonal projection of \( f \) on the line in \( \mathbb{C}_{\text{cl}}[G] \) spanned by \( \chi \)).

**Proof.** The map \( P_{\chi}(f) : V \to V \) is \( G \)-linear since

\[
\rho_s \circ P_{\chi}(f) \circ \rho_s^{-1} = \frac{1}{g} \sum_{t \in G} f(t)\rho_{st^{-1}s^{-1}}^{-1} = \frac{1}{g} \sum_{t \in G} f(sts^{-1})\rho_{(sts^{-1})^{-1}}^{-1} = P_{\chi}(f).
\]

Assume \( V \) irreducible. By Schur’s Lemma, there exists a unique \( \lambda \in \mathbb{C} \) such that \( P_{\chi}(f) = \lambda \text{id}_V \). Taking
traces, we get
\[ n \lambda = \text{tr}(\lambda \text{id}_V) = \text{tr}(P_\chi(f)) = \frac{1}{g} \sum_{t \in G} f(t) \chi(t^{-1}) = \frac{1}{g} \sum_{t \in G} f(t) \chi(t) = (f, \chi). \]
It follows that \( \lambda = \frac{(f, \chi)}{n} \).

**Remark 2.29.** Note that when \( \psi \) is a character of \( G \), \( P_\chi(\psi) \) can also be computed by the formula
\[ P_\chi(\psi) = \frac{1}{g} \sum_{t \in G} \psi(t) \rho_t^{-1} = \frac{1}{g} \sum_{t \in G} \psi(t^{-1}) \rho_t = \frac{1}{g} \sum_{t \in G} \overline{\psi}(t) \rho_t. \]
This formula is perhaps easier to compute (instead of computing the inverse of a group element or of the corresponding matrix, we only need to compute a complex conjugate).

**Remark 2.30.** Let \( G \) be a finite group and \( \psi_1, \psi_2 \in \mathbb{C}[G] \). Compare the definitions
\[ (\psi_1, \psi_2) \overset{\text{def}}{=} \frac{1}{g} \sum_{t \in G} \psi_1(t) \overline{\psi_2(t)}, \quad (\psi_1, \psi_2)_G \overset{\text{def}}{=} \frac{1}{g} \sum_{t \in G} \psi_1(t) \psi_2(t^{-1}). \]
We note that \( (\cdot | \cdot) \) defines a \( G \)-invariant inner product on \( \mathbb{C}[G] \). In the case where we work over the field of complex numbers (which we have been doing so far), letting \( \hat{\psi}(t) \overset{\text{def}}{=} \psi(t^{-1}) \), we have
\[ (\psi_1 | \psi_2) = \frac{1}{g} \sum_{t \in G} \psi_1(t) \overline{\psi_2(t)} = \frac{1}{g} \sum_{t \in G} \psi_1(t) \hat{\psi}(t^{-1}) = \left( \psi_1, \hat{\psi}_2 \right)_G. \]
The \( \mathbb{C} \)-bilinear map \( (\cdot, \cdot)_G \) generalizes \( (\cdot | \cdot) \) in the sense that it adapts very well to a representation theory over fields other than \( \mathbb{C} \), since for characters \( \psi \), we have \( \hat{\psi} = \psi \); clearly, this relationship still holds when \( \psi \) is a linear combination of characters with real coefficients (since \( \psi \mapsto \hat{\psi} \) is an \( \mathbb{R} \)-linear involution of \( \mathbb{C}[G] \)), so it holds in many useful cases.

This new map \( (\cdot, \cdot)_G \) shares many properties of \( (\cdot | \cdot) \); not all of them though, since the relationship \( \hat{\psi} = \psi \) only works in the \( \mathbb{R} \)-span of the characters, not in the \( \mathbb{C} \)-span. One sees for instance that Lemma 2.28 still holds (and the proof is even shorter, so it sounds like it is closer to being an argument meant to be proved with \( (\cdot, \cdot)_G \) rather than with \( (\cdot | \cdot) \)).

**Remark 2.31.** Let \( G \) be a finite group with irreducible representations \( \rho^1, \ldots, \rho^h \) and put each of these representations in matrix form \( \rho^k = (r^k_{ij}) \) where the matrices are all unitary, so that the relation \( (\rho^k)^{-1} = (\rho^k)^\dagger \) leads to \( r^k_{ij}(t^{-1}) = r^k_{ji}(t) \) for \( 1 \leq i, j \leq \deg \rho \), which means \( r^k_{ij} = r^k_{ji} \). In particular, by Corollary 1.34, we have
\[ \left( r^k_{ij_2}, r^k_{i_1, j_1} \right)_G = \left( r^k_{ij_2}, r^k_{i_1, j_1} \right)_G = \frac{1}{n_{k_1}} \delta_{i_1 j_2} \delta_{j_1 i_2} \delta_{j_1 k_2} \]
(where \( n_{k_1} = \deg \rho^k \)) so that the functions \( r^k_{ij} : G \to \mathbb{C} \) are **orthogonal** in \( \mathbb{C}[G] \). Note that there are
\[ \sum_{k=1}^h (\deg \rho^k)^2 = g \]
such functions according to Corollary 2.27, so since they are orthogonal, they actually span the space of all functions on \( G \). (Even better, if we normalize them by \( s^k_{ij} \overset{\text{def}}{=} \sqrt{n_{k_1}} r^k_{ij} \), then they become orthonormal.)

**Theorem 2.32.** The distinct irreducible characters \( \chi_1, \ldots, \chi_h \) of \( G \) form an orthonormal basis for \( \mathbb{C}[G] \).
Proof. We already proved that the irreducible characters are linearly independent in \( \mathbb{C}[G] \), so it suffices to prove that they generate \( \mathbb{C}[G] \). Let \( f \in \mathbb{C}[G] \) be orthogonal to all \( \chi_i \)'s. We know that for each irreducible representation \( \rho \) of \( G \) with character \( \chi \), \( P_\chi(f) = \frac{1}{n}(f, \chi) = 0 \). It follows that if we construct \( P_\chi(f) \) starting with any character of \( G \), after writing it as a sum of irreducible characters, we obtain \( P_\chi(f) = 0 \). Letting \( \chi \) denote the regular representation of \( G \), we obtain
\[
0 = P_\chi(f)(e_1) = \frac{1}{g} \sum_{t \in G} f(t) \rho_{t^{-1}}(e_1) = \frac{1}{g} \sum_{t \in G} f(t)e_{t^{-1}} \implies \forall t \in G, \ f(t) = 0.
\]

**Corollary 2.33.** The number of distinct irreducible representations of \( G \) is the number of conjugacy classes of \( G \).

**Proof.** Let \( C_1, \ldots, C_k \) be the \( k \) conjugacy classes of \( G \). If \( f_i : G \to \mathbb{C} \) is defined by
\[
f_i(s) = \begin{cases} 1 & \text{if } s \in C_i \\ 0 & \text{if not.} \end{cases}
\]
then \( \{f_1, \ldots, f_k\} \) is a basis of \( \mathbb{C}[G] \). It follows that \( k = h \). (We reserve \( h \) for the number of conjugacy classes/number of irreducible representations of \( G \) from now on.)

**Theorem 2.34.** (Column orthogonality relations) Recall that for \( s \in C_i \) where \( 1 \leq i \leq h \), we defined \( c_s \stackrel{\text{def}}{=} |C_i| \) and \( \chi_1, \ldots, \chi_h \) for the irreducible characters of \( G \). Then

(a) \( \sum_{i=1}^h \chi_i(s)\chi_i(s^{-1}) = \sum_{i=1}^h \chi_i(s)\chi_i(s) = \sum_{i=1}^h |\chi_i(s)|^2 = \frac{n}{c_s} \).

(b) For \( t \in G \) which is not conjugate to \( s \), we have
\[
\sum_{i=1}^h \chi_i(s)\chi_i(t^{-1}) = \sum_{i=1}^h \chi_i(s)\chi_i(t) = 0.
\]

**Proof.** Let \( f_j \) be defined as in the proof of Corollary 2.33, and pick \( t \in C_j \). By Theorem 2.32, we can write \( f_j = \sum_{i=1}^h (f_j | \chi_i) \chi_i \) where
\[
(f_j | \chi_i) = \frac{1}{g} \sum_{s \in G} f_j(s)\chi_i(s) = \frac{c_t}{g} \chi_j(s).
\]
It follows that for \( s \in G \),
\[
f_j(s) = \sum_{i=1}^h \left( \frac{c_t}{g} \chi_i(t) \right) \chi_i(s) = \frac{c_t}{g} \sum_{i=1}^h \chi_i(s)\chi_i(t)
\]
which gives (a) and (b) by the definition of \( f_j(s) \), whether we pick \( t \) conjugate to \( s \) or not.

**Definition 2.35.** Let \( \chi_1, \ldots, \chi_h \) be the characters of the irreducible representations \( W_1, \ldots, W_h \) of \( G \) with degrees \( n_1, \ldots, n_h \) and associated morphisms of groups \( \rho^i : G \to \text{GL}(W_i) \). If we decompose an arbitrary representation into a sum of irreducible representations, we can write \( m_i = (\chi_V, \chi_i) \) and write \( V \) as
\[
V = \bigoplus_{i=1}^h V_i, \quad V_i \simeq W_i^{\otimes m_i}
\]
where the notation $W^\oplus n$ denotes the direct sum of $n$ copies of the representation $W$; this can be done by producing an arbitrary direct sum decomposition of $V$ into irreducibles and collecting together those who are isomorphic. This direct sum decomposition of $V$ is called the **canonical decomposition** of $V$ and the $V_i$ are called the **canonical components** of $V$.

**Theorem 2.36.** The canonical decomposition of the representation $\rho : G \to \text{GL}(V)$ (of degree $n$ with character $\chi$) does not depend on the original decomposition into irreducible representations. The $G$-linear projection $\pi_i : V \to V$ with image $\text{im} \pi_i = V_i$ can be computed by

$$\pi_i = \frac{n_i}{g} \sum_{t \in G} \chi_i(t)\rho_t^{-1} = \frac{n_i}{g} \sum_{t \in G} \chi_i(t)\rho_t.$$  

**Proof.** We only need to show the first equality; the second one follows by re-arranging the sum. By Lemma 2.28 applied to the class function $f_i \overset{\text{def}}{=} n_i\chi_i$, if $W$ is an irreducible subrepresentation of $V$ with character $\chi_W$ of degree $n_W$, we have

$$\frac{n_i}{g} \sum_{t \in G} \chi_i(t)\rho_t^{-1}|_W = \frac{1}{g} \sum_{t \in G} f_i(t)\rho_t^{-1}|_W = P_{\chi_W}(f_i) = \frac{(f_i,\chi_W)}{n_W} = \frac{n_i(\chi_i,\chi_W)}{n_W}.$$  

Letting $V = \bigoplus_{j=1}^k W_j$ be a decomposition of $V$ into irreducible subrepresentations, we see that the $G$-linear map $\sum_{j=1}^k \sum_{t \in G} \chi_i(t)\rho_t^{-1}$ restricted to $W_j$ equals zero if $W_j \not\cong W_i$ and equals $\text{id}_{W_j}$ if $W_j \cong W_i$. It follows that $\text{im} \pi_i$ equals the sum of those $W_j$ isomorphic to $W_i$, and since $\pi_i$ does not depend on the chosen decomposition of $V$ into irreducible, neither does $V_i = \text{im} \pi_i$.

**Remark 2.37.** If we write $\mathbb{C}[G] = \bigoplus_{i=1}^k W_i^\oplus n_i$ in its canonical decomposition, we see that the class function $f_i = n_i\chi_i$ is the character of the canonical component associated to $\chi_i$.

### 2.4 Character tables

We begin this section with a few results which will serve as lemmas to compute character tables, which we then define and compute. Character tables form a major invariant of the class of finite groups. Although it does not characterize each group uniquely (two groups may have equivalent character tables), it is still a very strong tool and deserves being exploited; not only can it tell groups apart very often but it also helps understanding representations in general.

The last result of this section tells us how to work with irreducible representations obtained via the computation of the character table to explicitly decompose a representation given in matrix form into a direct sum of irreducible representations.

**Definition 2.38.** We begin by recalling a few concepts concerning group actions. Let $G$ be a finite group acting on a set $X$; we call $X$ equipped with the action $G \triangleleft X$ a $G$-set. A function $f : X \to Y$ between two $G$-sets satisfying $f(s \cdot x) = s \cdot f(x)$ is called a **morphism of $G$-sets**; this gives the collection of $G$-sets the structure of a category.

The **orbit** of a subset $A \subseteq X$ is the set $\text{Orb}_G(A) \overset{\text{def}}{=} GA = \{ s \cdot x \mid s \in G, x \in A \}$. In particular, the orbit of $x \in X$ is the set

$$\text{Orb}_G(x) \overset{\text{def}}{=} G\{ x \} = \{ s \cdot x \mid s \in G \}.$$  

Let $G \setminus X \overset{\text{def}}{=} \{ \text{Orb}_G(x) \mid x \in X \}$ denote the set of orbits of $G$ (in particular, if $H \leq G$ is a subgroup, $H \setminus G$ denotes the set of right cosets of $H$ in $G$ and the set of orbits under the left action of $H$ on $G$ given by $(h,g) \mapsto hg$. The two concepts agree, which explains why we chose this notation; we reserve the notation
$X/G$ for orbits of a **right action** $X \circ G$; c.f Section 3.6 for an example of when this is useful. A subset $S \subseteq X$ which contains precisely one element in each orbit of the action $G \circ X$ is called a **system of orbit representatives**. The **fixed point set** of $s \in G$ is defined by

$$X^s \overset{\text{def}}{=} \{ x \in X \mid s \cdot x = x \}. \tag{1}$$

The **stabilizer** of $x \in X$ is defined as

$$\text{Stab}_G(x) \overset{\text{def}}{=} \{ s \in G \mid s \cdot x = x \}. \tag{2}$$

More generally, if $A \subseteq X$, the **stabilizer** of $A$ is defined as

$$\text{Stab}_G(A) \overset{\text{def}}{=} \{ s \in G \mid s(A) = A \}. \tag{3}$$

(note that this means that the maps $s(-) : A \rightarrow A$ is bijective, but not necessarily equal to the identity map of $A$). In particular, $\text{Stab}_G(x) = \text{Stab}_G(\{ x \})$.

Note that in general, the subgroup $\text{Stab}_G(x) \leq G$ is not a normal subgroup since by definition, for $s \in G$ and $A \subseteq X$,

$$s \text{Stab}_G(A)s^{-1} = \{ sts^{-1} \in G \mid t(sA) = A \} = \{ t \in G \mid s^{-1}ts(A) = (A) \} = \text{Stab}_G(s(A)).$$

To define the **index** $[G : H]$ of a subgroup $H$ of $G$, note that the map $gH \mapsto (gH)^{-1} = Hg^{-1}$ is a bijection between the set of left cosets $G/H$ and the set of right cosets $H \backslash G$. The cardinality of any two of these sets is the definition of $[G : H]$. By Lagrange’s theorem, when $G$ is finite, we have $|H||G : H| = |G|$, hence $|G : H| = |G|/|H|$.

We say that a group action is **transitive** if $G$ has only one orbit; in other words, $G$ is transitive if for every $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$.

A group action $G$ on the set $X$ induces a natural diagonal action on $X^2 \overset{\text{def}}{=} X \times X$ via $g \cdot (x, x') \overset{\text{def}}{=} (g \cdot x, g \cdot x')$. If we denote by $\Delta : X \rightarrow X \times X$ the map $x \mapsto (x, x)$, since $g \cdot \Delta(x) = \Delta(g \cdot x, x))$, we see that $\text{Orb}_G(\Delta(x)) = \Delta(\text{Orb}_G(x))$. (In particular, if $|X| > 1$, the action $G \circ X^2$ is never transitive.) The action of $G$ on $X$ is said to be **doubly transitive** if the action of $G$ on $X \times X \setminus \Delta(X)$ is transitive, namely for every $(x, y), (x', y') \in X \times X \setminus \Delta(X)$, there exists $g \in G$ with $g \cdot (x, y) = (x', y')$.

More generally, the action of $G$ on $X$ is said to be $k$-**transitive** if $k \leq |X|$ and the diagonal action on $X^k$ (given by $g \cdot (x_1, \ldots, x_k) \overset{\text{def}}{=} (g \cdot x_1, \ldots, g \cdot x_k)$) has the minimal possible number of orbits for an action on $X (X^k$ has to have some minimal number of orbits; for example, when $k = 2$, those are $\Delta(X)$ and $X^2 \setminus \Delta(X)$, and the minimal number of orbits is always achieved by the permutation group of $X$). Phrased differently, $G \circ X$ is $k$-transitive if for any $1 \leq \ell \leq k$, any two ordered $\ell$-tuples $(x_1, \ldots, x_\ell)$ and $(y_1, \ldots, y_\ell)$ of distinct elements of $X$ (i.e. where $x_i \neq x_j$ and $y_i \neq y_j$ for $1 \leq i \neq j \leq \ell$) can be mapped onto another via multiplication by some $s \in G$, namely $s \cdot (x_1, \ldots, x_\ell) = (y_1, \ldots, y_\ell)$. It follows that a $k$-transitive action is also $\ell$-transitive for any $1 \leq \ell \leq k$.

**Theorem 2.39.** (Orbit-Stabilizer Theorem) Let the (not necessarily finite) group $G$ act on the (not necessarily finite) set $X$ and let $x \in X$. Then there is a natural bijection of sets $G/\text{Stab}_G(x) \simeq \text{Orb}_G(x)$ (in the category of $G$-sets), which implies $|G : \text{Stab}_G(x)| = |\text{Orb}_G(x)|$. In particular, If $G$ is a finite group of order $g$, then $|\text{Stab}_G(x)||\text{Orb}_G(x)| = g$.

**Proof.** Consider the map $\varphi : G \rightarrow \text{Orb}_G(x)$ defined by $s \mapsto s \cdot x$. This map is surjective by definition. If $s, t \in G$ satisfy $s \cdot x = t \cdot x$, then $s^{-1}t \in \text{Stab}_G(x)$, so the map $\tilde{\varphi} : G/\text{Stab}_G(x) \rightarrow \text{Orb}_G(x)$ defined by the same formula defines a transitive group action $G/\text{Stab}_G(x) \circ \text{Orb}_G(x)$, which means that $\tilde{\varphi}$ is a bijection, namely $|G : \text{Stab}_G(x)| = |G/\text{Stab}_G(x)| = |\text{Orb}_G(x)|$. For naturality, it suffices to see that $\tilde{\varphi}$ is a morphism of $G$-sets, which is equivalent to the identity $s \cdot (t \cdot x) = (st) \cdot x$. When $G$ is finite, we have

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**Note:** The above text provides a detailed explanation of the concepts of group actions, orbit-stabilizer theorems, and related mathematical structures. It is intended to clarify the definitions and theorems presented in the original text, ensuring a smooth understanding of the material. The notation and terminology used are consistent with standard mathematical literature on group theory and related fields.
Chapter 2

\[ |G : \text{Stab}_G(x)| = |G|\div |\text{Stab}_G(x)| = g\div |\text{Stab}_G(x)|, \text{ which proves the result.} \]

**Lemma 2.40.** (Burnside’s Lemma) Let the finite group \( G \) of order \( g \) act on the finite set \( X \). Then

\[ |G\setminus X| = \frac{1}{g} \sum_{s \in G} |X^s|. \]

**Proof.** We have

\[ \sum_{s \in G} |X^s| = \{|(s, x) \in G \times X \mid s \cdot x = x\}| = \sum_{x \in X} |\text{Stab}_G(x)|. \]

By the Orbit-Stabilizer theorem, we have \(|\text{Stab}_G(x)||\text{Orb}_G(x)| = g\), hence by writing \( X \) as the disjoint union of its orbits, we have

\[
\begin{align*}
\sum_{x \in X} |\text{Stab}_G(x)| &= \sum_{x \in X} \frac{g}{|\text{Orb}_G(x)|} \\
&= g \sum_{x \in X} \frac{1}{|\text{Orb}_G(x)|} \\
&= g \sum_{\text{Orb}_G(x) \subseteq G \setminus X} \left( \sum_{y \in \text{Orb}_G(x)} \frac{1}{|\text{Orb}_G(x)|} \right) \\
&= g \sum_{\text{Orb}_G(x) \subseteq G \setminus X} 1 \\
&= g |G\setminus X|.
\end{align*}
\]

**Proposition 2.41.** If \( V \) is a permutation representation with character \( \chi \) associated to the action of the finite group \( G \) on a finite set \( X \), then \( \chi(s) = |X^s| \) is the number of elements in \( X \) fixed by \( s \in G \).

**Proof.** We have \( \rho : G \to \GL(V) \) where \( V \) is the free vector space on the set \( X \) with basis \( \{e_x\}_{x \in X} \) and \( s \in G \) satisfies \( \rho_s(e_x) = e_{s \cdot x} \) for all \( x \in X \). Computing the trace over this basis, it follows that \( x \in X \) contributes to either 0 or 1 to \( \chi(s) \), depending on \( e_x = e_{s \cdot x} \) or not. Summing over all \( x \in X \) gives the result.

**Proposition 2.42.** Let the finite group \( G \) act on the finite set \( X \). Consider the corresponding permutation representation \( V \) with character \( \chi \) and denote the trivial representation of \( G \) by \( \chi_1 \). Then \( |G\setminus X| = (\chi, \chi_1) \) and there exists a unique character \( \theta \) of \( G \) such that \( \theta + |G\setminus X| \chi_1 = \chi \) and \( (\theta, \chi_1) = 0 \). This character is equal to zero (i.e. the trace of the zero representation) if and only if \( G \) acts trivially on \( X \) (i.e. \( s \cdot x = x \) for all \( (s, x) \in G \times X \)). This character is called the standard representation associated to \( \chi \).

**Proof.** This follows directly from Burnside’s Lemma since \( \chi(s) = |X^s| \), hence

\[ (\chi, \chi_1) = \frac{1}{g} \sum_{s \in G} \chi(s) \overline{\chi_1}(s) = \frac{1}{\text{ord}(G)} \sum_{s \in G} |X^s| = |G\setminus X| \]

The character \( \theta \) is well-defined. Writing \( \chi \) as a sum of irreducible characters, we see that \( \theta(1) \neq 0 \) if and only if \( |G\setminus X| < |X| \), which completes the proof.

**Remark 2.43.** There is nothing wrong in the unique morphism of groups \( \rho : G \to \GL(0) \) where all morphisms have zero trace according to Definition 1.31 since the composition \( \Hom_{\mathbb{C}}(V, V) \simeq V^* \otimes V \xrightarrow{\theta} \mathbb{C} \)
defines the trace map on $V = 0$ as the zero map $0 = V \to \mathbb{C}$. If one is uneasy with dealing with a statement such as “the trace of the zero representation”, one can interpret the statement of Proposition 2.42 differently. Namely, one can consider $\theta$ as a class function in full generality, and it will become the character of a positive-dimensional vector space if and only if the action $G \ltimes X$ is non-trivial, i.e. if there exists at least one pair $(s, x) \in G \times X$ such that $s \cdot x \neq x$.

However, there is a theoretical interest in considering the zero representation as a representation since it turns the set of isomorphism classes of representations of $G$ over $\mathbb{C}$ into a commutative monoid under the operation of direct sum; the zero representation is the identity of this monoid since $0 \oplus V \simeq V \simeq V \oplus 0$. By Corollary 2.21 and Proposition 2.10, this is isomorphic to the monoid of characters of $G$ under addition. By Theorem 2.32, if $\text{Rep}_C(G)/\sim$ denotes this monoid (the $/ \sim$ stands for “isomorphism classes of”) and $\text{Gr}$ denotes the operation of taking the Grothendieck group of a commutative monoid, then

$$\mathbb{C} \otimes_\mathbb{Z} \text{Gr}(\text{Rep}_C(G)/\sim) \simeq \mathbb{C}_{\text{cl}}[G].$$

(Interpreting $\text{Rep}_C(G)/\sim$ as the $\mathbb{N}$-span of the irreducible characters sitting inside $\mathbb{C}_{\text{cl}}[G]$, its Grothendieck group corresponds to the $\mathbb{Z}$-span of the irreducible characters, making the result trivial.)

**Lemma 2.44.** Let $G \ltimes X$ and let $\chi$ be the character corresponding to the associated permutation representation. If $G$ acts diagonally on $X^k$ (c.f. Definition 2.38), then

$$\frac{1}{g} \sum_{s \in G} \chi(s)^k = |G\backslash X^k|.$$  

**Proof:** If $s \in G$ has $|X^s|$ fixed points in $X$, the corresponding diagonal action $G \ltimes X^k$ is such that $|(X^s)^k| = |X^s|^k$; this is because $(x_1, \ldots, x_k) \in (X^k)^s$ if and only if $x_1, \ldots, x_k \in X^s$. Since $\chi(s) = |X^s|$ by Proposition 2.41, the result follows by applying Burnside’s Lemma to the diagonal action $G \ltimes X^k$.

**Remark 2.45.** Lemma 2.44 also follows from the fact that $\chi^k$ is the character of the $k$th power of $\chi$ and the permutation representation associated to $G \ltimes X^k$ happens to be the $k$-fold tensor product of the permutation representation associated to $G \ltimes X$.

**Theorem 2.46.** Let $X$ be a finite set with $|X| > 1$ and let the finite group $G$ act transitively on $X$ (so that $|G\backslash X| = 1$). Denote the associated representation by $V$ and its character by $\chi$. Set $\theta \overset{\text{def}}{=} \chi - \chi_1$ (c.f. Proposition 2.42). Then the following are equivalent:

(i) $G$ acts doubly transitively on $X$.

(ii) $|G\backslash X^2| = 2$, i.e. the diagonal action $G \ltimes X^2$ has the two orbits $\Delta(X)$ and $X^2 \setminus \Delta(X)$

(iii) $(\chi^2, 1) = 2$.

(iv) The representation $\theta$ is irreducible.

**Proof:** Statements (i) and (ii) are trivially equivalent since $\Delta(X)$ and $X^2 \setminus \Delta(X)$ have to be a disjoint union of orbits of the diagonal action $G \ltimes X^2$. As for (iii), since $\chi(s) = |X^s|$, it suffices to see that by Lemma 2.44,

$$(\chi^2, 1) = \frac{1}{g} \sum_{s \in G} \chi(s)^2 = \frac{1}{g} \sum_{s \in G} |X^s|^2 = |G\backslash X^2|.$$  

which proves that (ii) and (iii) are equivalent.

For (iv), note that $\theta$ is irreducible if and only if $(\theta, \theta) = 1$, so we compute : since $\chi = \chi$ and $\chi_1 = \chi_1$, we
have
\[(\theta, \theta) = (\chi - \chi_1, \chi - \chi_1) = (\chi, \chi) - 2(\chi, \chi_1) + (\chi_1, \chi_1) = (\chi^2, 1) - 1.\]
(Note that by Proposition 2.42, \((\chi, \chi_1) = 1\) by assumption that \(G\) is transitive.) Therefore \((\theta, \theta) = 1\) if and only if \((\chi^2, 1) = 2\), i.e. (iii) and (iv) are equivalent.

Corollary 2.47. Let \(n \geq 1\) be an integer, and consider the standard action of \(S_n\) on \(X \defeq \{1, \cdots, n\}\). Denote its character by \(\chi\) and the trivial representation of \(S_n\) by \(\chi_1\). Then the standard representation \(\theta \defeq \chi - \chi_1\) is irreducible.

**Proof.** This follows since \(S_n\) acts doubly transitively on \(\{1, \cdots, n\}\); the action \(S_n \times X\) has only one orbit since it is transitive, hence \(|S_n\backslash X| = 1\), so that the standard representation \(\chi - \chi_1\) indeed equals \(\theta\). (In fact, \(S_n\) even acts \(n\)-transitively on \(\{1, \cdots, n\}\) by its very definition since given any two \(n\)-tuples \((x_1, \cdots, x_n)\) and \((y_1, \cdots, y_n)\) (i.e. two orderings of the elements of \(\{1, \cdots, n\}\)), there is a unique permutation of \(\{1, \cdots, n\}\) which maps \(x_i\) to \(y_i\) for all \(i = 1, \cdots, n\) (and it is defined by this formula).)

We now have enough information to begin computing a few character tables, a notion which we now define.

**Definition 2.48.** Let \(G\) be a finite group. Assume the irreducible characters of \(G\) are given by \(\chi_1, \cdots, \chi_h\) and the conjugacy classes of \(G\) are denoted by \(C_1, \cdots, C_h\). A **character table** of \(G\) is an \(h \times h\) matrix whose \((i, j)\)-entry equals \(\chi_i(s)\) for some \(s \in C_j\) (i.e. any \(s \in C_j\)). When we want to omit defining \(s \in C_j\) explicitly, we write \(\chi_i(C_j) \defeq \chi_i(s)\). By re-indexing the characters and the conjugacy classes, one obtains different character tables; the set of all such possible character tables is denoted by \(\text{Ch}(G)\). When we are only interested in listing the irreducible characters of \(G\), we write them down in a table (see Example 2.51 for instance) and speak of “the” character table of \(G\).

**Remark 2.49.** If \(\varphi\) is an automorphism of \(G\) and \(\rho : G \to \text{GL}(V)\) is a representation of \(G\), then \(\rho \circ \varphi : G \to \text{GL}(V)\) is also a representation. This induces a right action of the automorphism group \(\text{Aut}(G)\) on \(\text{Rep}_C(G)\) via \(\rho^\varphi \defeq \rho \circ \varphi\) (right actions are often denoted using exponential notation in order to set them on the right of the element they act on). Note that if \(\rho\) is irreducible, so is \(\rho^\varphi\); one can even see that given \(v \in V \setminus \{0\}\), the vector space \(V\) is the \(C\)-span of \(\{\rho(s)\,|\,s \in G\}\) and this set of generators remains unchanged if we replace \(\rho\) by \(\rho^\varphi\) (it only permutes that set).

If \(\chi\) is the corresponding character of the representation \(\rho\), then the character of \(\rho^\varphi\) equals \(\chi^\varphi \defeq \chi \circ \varphi :\)
\[\chi^\varphi(s) = \text{tr}(\rho^\varphi_s) = \text{tr}(\rho_{\varphi(s)}) = \chi(\varphi(s)),\]

hence \(\text{Aut}(G)\) acts on the right on the set of irreducible characters of \(G\); since inner automorphisms of \(G\) act trivially on characters, this induces a right action of the **outer automorphism group** \(\text{Out}(G) \defeq \text{Aut}(G)/\text{Inn}(G)\) on the set of irreducible characters of \(G\), and therefore on \(\text{Ch}(G)\) via the formula
\[(\chi_i(C_j))^\varphi \defeq (\chi_i^\varphi(C_j)).\]

This action can be helpful to produce new irreducible characters out of already known ones; in particular, note that abelian groups \(G\) (written additively) admit the outer automorphism \(s \mapsto -s\) (abelian groups are precisely those groups for which \(\text{Inn}(G)\) is trivial), and this outer automorphism is non-trivial unless \(G\) is an abelian elementary 2-group (every non-zero element has order 2).

**Proposition 2.50.** Let \(G\) be a finite group. Assume \(\chi, \chi'\) are two irreducible characters of \(G\) where \(\deg \chi = 1\). Then \(\chi \chi'\) is an irreducible character of \(G\). In particular, the irreducible characters of degree 1 form a **group** under multiplication which acts on the set of irreducible characters of \(G\). (When \(G\) is abelian, this group is equal to \(\hat{G}\), the Pontryagin dual group of \(G\)).
**Proof.** Recall that a character of degree \( n \) takes values which are sums of \( n \)th roots of unity by Theorem 2.11; in particular, \( \chi \) takes values in the set of roots of unity, so for any \( s \in G, |\chi(s)| = 1 \). Therefore, by Corollary 2.20, we see that \( \chi \chi' \) is irreducible since

\[
(\chi \chi', \chi \chi') = \frac{1}{g} \sum_{s \in G} |\chi \chi'(s)|^2 = \frac{1}{g} \sum_{s \in G} |\chi(s)|^2 |\chi'(s)|^2 = \frac{1}{g} \sum_{s \in G} |\chi'(s)|^2 = (\chi', \chi') = 1.
\]

**Example 2.51.** Consider the group \( S_3 \). One can easily see that \( S_3 \) admits 3 conjugacy classes, namely the identity permutation, the even permutations and the odd ones. Therefore, \( S_3 \) admits precisely 3 irreducible characters. We already know three irreducible representations of \( S_3 \), namely the trivial and the sign representation (both of degree 1, hence irreducible) and the standard representation of \( S_3 \), which is irreducible of degree 2.

We can use this information to build a character table of \( S_3 \) : indicating each conjugacy class by representing it via one of its elements, we obtain

<table>
<thead>
<tr>
<th>( c_s )</th>
<th>1</th>
<th>3</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_3 )</td>
<td>(1)</td>
<td>(12)</td>
<td>(123)</td>
</tr>
<tr>
<td>trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>sign</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>standard</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

**Example 2.52.** Consider \( G = S_4 \). Recall that in general, for the group \( S_n \), the conjugacy class of a permutation is completely determined by its cycle structure. More precisely, one can show that the conjugacy classes are in one-to-one correspondence with partitions of the integer \( n \). This is better understood from the following identity: let \( (a_1 \ a_2 \ \ldots \ a_m) \) denote an \( m \)-cycle in \( S_n \) and \( \sigma \in S_n \). Then

\[
\sigma(a_1 \ a_2 \ \cdots \ a_m)\sigma^{-1} = (\sigma(a_1) \ \sigma(a_2) \ \cdots \ \sigma(a_m))
\]

(This could have also quickly determined the conjugacy classes of \( S_3 \) in Example 2.51.) It follows from this that the conjugacy class of a permutation depends only on the length of its disjoint cycles (this is what we call the cycle structure of a permutation). For instance, we have the following partitions of the integer 4 (we write the cycles very explicitly to explain the correspondence): letting \( C(\sigma) \) denote the conjugacy class of the permutation \( \sigma \in S_4 \), we have

\[
\begin{align*}
1 + 1 + 1 + 1 & \iff C((1)(2)(3)(4)) \\
2 + 1 + 1 & \iff C((12)(3)(4)) \\
3 + 1 & \iff C((123)(1)) \\
4 & \iff C((1234)) \\
2 + 2 & \iff C((12)(34))
\end{align*}
\]

We can now begin to build a character table. We indicate the number of elements in each conjugacy class in a row above to help in computations. Using Proposition 2.41 and Corollary 2.47, we obtain the first, second and third irreducible representations as the trivial, sign and standard representations of \( G \), which gives the following:

<table>
<thead>
<tr>
<th>( c_s )</th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>6</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_4 )</td>
<td>(1)</td>
<td>(12)</td>
<td>(123)</td>
<td>(1234)</td>
<td>(12)(34)</td>
</tr>
<tr>
<td>( \chi_1 ), trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 ), sign</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 ), standard</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( \chi_4 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \chi_5 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
One sees by Proposition 2.50 that $\chi_4 \overset{\text{def}}{=} \chi_2 \chi_3$ are irreducible. Since there is only one representation left, we can use the fact that $g = 4! = 24$ is the sum of the squares of the degrees of $\chi_1, \cdots, \chi_5$ (c.f. Corollary 2.27) and the column orthogonality relations (c.f. Theorem 2.34) to fill in the last row. We get the full table:

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>1</th>
<th>6</th>
<th>8</th>
<th>6</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_4$</td>
<td>(1)</td>
<td>(12)</td>
<td>(123)</td>
<td>(1234)</td>
<td>(12)(34)</td>
</tr>
<tr>
<td>$\chi_1$, trivial</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$, sign</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$, standard</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

2.5 Characters of abelian groups

**Theorem 2.53.** Let $G$ be a finite group of order $g$. Then $G$ is abelian $\iff$ all its irreducible characters have degree 1.

**Proof.** Let $n_1, \cdots, n_h$ the degrees of the irreducible representations. Recall that $h$ is the number of irreducible representations of $G$ and the number of conjugacy classes of $G$. By Corollary 2.27, $\sum_{i=1}^{h} n_i^2 = g$, so if $n_i = 1$ for $1 \leq i \leq h$, this means $h = g$; in particular, $G$ has $g$ conjugacy classes, so it must be abelian.

Conversely, if $G$ is abelian, $h = g$, hence $\sum_{i=1}^{h} n_i^2 = g$ implies that $n_i = 1$ for each $1 \leq i \leq h = g$.

**Corollary 2.54.** Assume $A \leq G$ is an abelian subgroup. Then every irreducible representation of $G$ has degree at most $|G : A| = \frac{\text{ord}(G)}{\text{ord}(A)}$.

**Proof.** Let $\rho : G \to \text{GL}(V)$ be an irreducible representation of $G$, hence giving a representation $\rho_A \overset{\text{def}}{=} \rho|_A$ of the subgroup $A$. Take an irreducible subrepresentation $W$ of $\rho_A : A \to \text{GL}(V)$, so that $\dim W = 1$ by Theorem 2.53. Put

$$V' \overset{\text{def}}{=} \sum_{s \in G} \rho_s(W) \subseteq V,$$

Since $V' \neq 0$ is $G$-stable and $V$ is an irreducible representation of $G$, we must have $V' = V$. For $s \in G$ and $t \in A$, we have

$$\rho_{st}(W) = \rho_s(\rho_t(W)) = \rho_s(W),$$

hence the image of $\rho_{st}$ depends only on which coset of $A$ the element $st$ is in $G$. If we write $W = \langle w \rangle_G$ and take a set of representatives $R$ for the left cosets of $G/A$, then $V$ is generated by $\{\rho_s(w) \mid s \in R\}$, hence $\dim V \leq |R| = |G : A|$.

**Remark 2.55.** In the previous proof, we could be tempted to write $H = \{g \in G \mid \rho_g(W) = W\}$ with $A \leq H \leq G$, take a coset of representatives $R$ for $G/H$ and conclude that since $V$ is spanned by $\{\rho_r(w) \mid r \in R\}$, that $\dim V = |R| = |G : H|$ (which divides $|G : A|$) and get the stronger result “the degree divides $|G : A|$”. First of all, this is false since there are counter-examples; any non-abelian group of order $pq$ where $p, q$ are distinct primes is a counter-example because they have irreducible representations of degree $> 1$ by Theorem 2.53, but dividing $p$ and $q$ implies being equal to 1). Second of all, the flaw in the proof is when we assume $\dim V = |R|$; the spaces $\rho_r(W)$ are pairwise linearly independent, but not necessarily linearly independent. So this argument only works when we can be sure that $|G : H| = 2$, e.g. when $|G : A| = 2$; but in this case we don’t learn anything new (because $\dim V$ dividing $|G : A|$ and $\dim V \leq |G : A|$ are
equivalent), so it is not interesting.

**Example 2.56.** The dihedral group $D_n$ contains an abelian group of index 2 (the cyclic subgroup of order $n$), hence all irreducible representations have degree $\leq 2$. We will compute the character table of $D_n$ in Example 2.71.

### 2.6 Character table of a product of two groups

Recall that given two finite groups $G_1, G_2$ of order $g_1$ and $g_2$ respectively, one can form their (external) direct product $G_1 \times G_2 \overset{\text{def}}{=} \{(s_1, s_2) \mid s_i \in G_i\}$ with the operation $(s_1, s_2)(t_1, t_2) \overset{\text{def}}{=} (s_1 t_1, s_2 t_2)$. The group $G = G_1 \times G_2$ has order $g = g_1 g_2$, we have canonical inclusions $G_i \hookrightarrow G_1 \times G_2$ given by $s_1 \mapsto (s_1, 1)$ and $s_2 \mapsto (1, s_2)$. If we identify $G_i$ with its image under this inclusion, then the elements of $G_1$ commute with the elements of $G_2$.

Conversely, given a group $G$ which contains two subgroups $G_1, G_2$ satisfying $G_1 G_2 = G$ and $G_1 \cap G_2 = \{1\}$ where the elements of $G_1$ commute with the elements of $G_2$, then $G \cong G_1 \times G_2$; this is because in this case, each element of $G$ can be written uniquely in the form $s = s_1 s_2$ where $s_i \in G_i$. (In this case, we say that $G$ is the **internal direct product** of $G_1$ and $G_2$.) Using the results of this section, when we recognize a group $G$ as the internal direct product of two subgroups $G_1$ and $G_2$, we can deduce the character table of $G$ by the character tables of $G_1$ and $G_2$.

**Theorem 2.57.** Let $G = G_1 \times G_2$ and $\rho^i : G_i \to \text{GL}(V_i)$ be two representations ($i = 1, 2$) with character $\chi^i$. The box product representation $\rho^1 \boxtimes \rho^2$ has character $\chi^1 \boxtimes \chi^2$, called the **box product** of $\chi^1$ and $\chi^2$, which is given by

\[
(\chi^1 \boxtimes \chi^2)(s_1, s_2) = \chi^1(s_1) \chi^2(s_2).
\]

The representation $\rho^1 \boxtimes \rho^2$ is irreducible if and only if $\rho^1$ and $\rho^2$ are irreducible. Each irreducible representation of $G$ is of the form $\rho^1 \boxtimes \rho^2$ for some irreducible representations of $G_1$ and $G_2$. (c.f. Definition 1.22).

**Proof.** The formula for the character $\chi$ is computed similarly as the character of the tensor product of two representations over $G$ : fix a basis of $V_i$ over which $\rho^i$ is diagonal (c.f. Theorem 2.11), say $\{v_{i1}, \ldots, v_{in_i}\}$. Then $\{v_{j1} \boxtimes v_{j2} \mid 1 \leq j_i \leq n_i, i = 1, 2\}$ is a basis of $V_1 \boxtimes V_2$ and $\rho$ has eigenvalue $\lambda^i_j$ at $v_{j_i}$. We have

\[
(\rho^1 \boxtimes \rho^2)(v_{j1} \boxtimes v_{j2}) = \rho^1(v_{j1}) \boxtimes \rho^2(v_{j2}) = \lambda^1_{j_1} \lambda^2_{j_2} (v_{j_1} \boxtimes v_{j_2}),
\]

so that $\rho$ is diagonalized over this basis. Summing all the eigenvalues shows that $\chi^1 \boxtimes \chi^2$ is indeed the character of $\rho^1 \boxtimes \rho^2$.

The second statement holds because

\[
(\chi^1 \boxtimes \chi^2, \chi^1 \boxtimes \chi^2) = \frac{1}{g} \sum_{s_1 \in G_1} \sum_{s_2 \in G_2} |(\chi^1 \boxtimes \chi^2)(s_1, s_2)|^2 = \frac{1}{g_1 g_2} \sum_{s_1 \in G_1} \sum_{s_2 \in G_2} |\chi^1(s_1)|^2 |\chi^2(s_2)|^2 = \left(\frac{1}{g_1} \sum_{s_1 \in G_1} |\chi^1(s_1)|^2\right) \left(\frac{1}{g_2} \sum_{s_2 \in G_2} |\chi^2(s_2)|^2\right) = (\chi^1, \chi^1)(\chi^2, \chi^2).
\]

For the last statement, it suffices to show that if $f \in C[G_1 \times G_2]$ is orthogonal to all characters of the form $\chi^1 \boxtimes \chi^2$ where $\chi^1$ (resp. $\chi^2$) is an irreducible representation of $G_1$ (resp. $G_2$), then $f = 0$. We have

\[
(f, \chi^1 \boxtimes \chi^2) = \frac{1}{g} \sum_{s_1 \in G_1} \sum_{s_2 \in G_2} f(s_1, s_2) \overline{\chi^1(s_1)} \chi^2(s_2) = 0.
\]
Define \( f' : G_2 \to \mathbb{C} \) by
\[
f'_i(s_2) \overset{\text{def}}{=} \frac{1}{g_1} \sum_{s_1 \in G_1} f(s_1, s_2) \chi_i^1(s_1) \in \mathbb{C}_1[G_2].
\]
Since \( f \) is a class function, \( f' \) is also; since the \( \chi_j^2 \)'s form a basis of \( \mathbb{C}_1[G_2] \) and
\[
\frac{1}{g_2} \sum_{s_2 \in G_2} f'_i(s_2) \chi_j^2(s_2) = (f, \chi_i^1 \boxtimes \chi_j^2) = 0,
\]
we obtain \( f'_i = 0 \); since the \( \chi_j^1 \)'s form a basis of \( \mathbb{C}_1[G_1] \), we have
\[
\frac{1}{g_1} \sum_{s_1 \in G_1} f(s_1, s_2) \chi_i^1(s_1) = f'_i(s_2) = 0 \implies \forall (s_1, s_2) \in G_1 \times G_2, \quad f(s_1, s_2) = 0.
\]

**Definition 2.58.** Let \( A \in \text{Mat}_{m \times m}(\mathbb{C}) \) and \( B \in \text{Mat}_{n \times n}(\mathbb{C}) \). The **Kronecker product** of \( A \) and \( B \) is the matrix \( A \otimes B \in \text{Mat}_{mn \times mn}(\mathbb{C}) \) defined by
\[
A \otimes B \overset{\text{def}}{=} \begin{bmatrix}
a_{11}B & \cdots & a_{1m}B \\
\vdots & \ddots & \vdots \\
a_{m1}B & \cdots & a_{mm}B
\end{bmatrix}
\]
More explicitly in terms of coefficients, for \( 1 \leq i, k \leq m \) and \( 1 \leq j, \ell \leq n \), we have
\[
(A \otimes B)_{(i-1)n+j, (k-1)n+\ell} = a_{ik}b_{j\ell}.
\]

**Proposition 2.59.** Let \( G_1 \) (resp. \( G_2 \)) be a finite group with character table \( A_1 = (\chi_i^1(C_k^1)) \) (resp. \( A_2 = (\chi_j^2(C_k^2)) \)). Then \( A_1 \otimes A_2 \) is a character table of \( G_1 \times G_2 \).

**Proof.** First note that the conjugacy classes of \( G_1 \times G_2 \) are of the form \( C_k^1 \times C_k^2 \) since for \( (s_1, s_2) \in G_1 \times G_2 \) and \( (t_1, t_2) \in C_k^1 \times C_k^2 \),
\[
(s_1, s_2)(t_1, t_2)(s_1, s_2)^{-1} = (s_1t_1s_1^{-1}, s_2t_2s_2^{-1}) \in C_k^1 \times C_k^2.
\]
It follows that
\[
(A_1 \otimes A_2)_{((i-1)n+j, (k-1)n+\ell)} = \chi_i^1(C_k^1) \chi_j^2(C_k^2) = (\chi_i^1 \otimes \chi_j^2)(C_k^1 \times C_k^2).
\]
The rows of \( A_1 \otimes A_2 \) form all the irreducible characters of \( G \) by Theorem 2.57, so we are done.

### 2.7 Lifted and dropped representations

This section is meant to put into correspondence the representations of \( G/K \) with a subset of those of \( G \) when \( K \leq G \) is a normal subgroup.

**Definition 2.60.** Let \( K \leq G \) be a normal subgroup of the finite group \( G \). Given a representation \( \rho : G/K \to \text{GL}(V) \), the **lift** of \( \rho \) to \( G \) is the representation \( L(\rho) \overset{\text{def}}{=} \rho \circ \pi_K : G \to \text{GL}(V) \) defined by \( L(\rho)_s \overset{\text{def}}{=} \rho_sK \). If \( \chi \) is the character of \( \rho \), we write \( L(\chi) \) for the character of \( L(\rho) \).

Conversely, if \( \eta : G \to \text{GL}(V) \) is a representation of \( G \) such that for all \( s \in K \), we have \( \eta_s = \text{id}_V \), we obtain a morphism of groups \( D(\eta) : G/K \to \text{GL}(V) \) by passing to the quotient, i.e. a representation of \( G/K \) called the **drop** of \( \eta \) along \( K \). Analogously, if \( \psi \) is the character of \( \eta \), we write \( D(\psi) \) for the character of \( D(\eta) \).
**Theorem 2.61.** The lift and drop are inverse constructions of each other in the following sense: the maps $L(-)$ and $D(-)$ are bijections between the sets

$$\text{Rep}_C(G/K) \leftrightarrow \{ \rho \in \text{Rep}_C(G) \mid \ker \rho \supseteq K \}.$$ 

In other words, $D(L(\rho)) = \rho$ and $L(D(\eta)) = \eta$. Let $\rho$ be a representation of $G/K$ with character $\chi$ and $\eta$ be a representation of $G$ with character $\psi$. For $s \in G$, we have the formulas

$$L(\chi)(s) = \chi(sK), \quad D(\psi)(sK) = \psi(s).$$

Furthermore, $L(-)$ and $D(-)$ restrict to a bijection between the irreducible representations in each set.

**Proof.** The bijectivity is given by the quotient map correspondence for groups. As for characters, we are evaluating the trace of the same matrices, so it remains unchanged. For irreducibility, if $\rho : G/K \to \text{GL}(V)$ is a representation, then $V$ is irreducible if and only if for any $v \in V \setminus \{0\}$, the set

$$\{ \rho(sK) v \mid sK \in G/K \}$$

spans $V$; this set is also equal to $\{ L(\rho)(s)(v) \mid s \in G \}$ since for $t \in K$, $L(\rho)(t)v = \rho(tK)(v) = \rho_K(v) = v$. Since those two sets of generators are always equal, we deduce that $\rho$ is irreducible if and only if $L(\rho)$ is, which is what was required to prove.

**Example 2.62.** (Products of groups) Let $G = G_1 \times G_2$ be two groups. One sees that the subgroups $G_1, G_2 \trianglelefteq G$ are normal, so if $\rho^1$ (resp. $\rho^2$) is a representation of $G_1$ (resp. $G_2$), then

$$\rho^1 \boxtimes \rho^2 \simeq L(\rho^1) \otimes L(\rho^2)$$

(note that the first tensor product is the one between representations of $G_1$ and $G_2$, while the second one is between representations of $G$ since $L(\rho^i)$ is lifted from $G_i$ to $G$). In other words, one can produce all irreducible representations of $G$ by lifting those of $G_1$ and $G_2$, and then tensoring them together (had we not known that we can tensor them together without lifting to begin with).

**Remark 2.63.** It is easy to detect irreducible lifted representations via the character table when one knows the normal subgroups of a group. Normal subgroups are unions of conjugacy classes, so if a character in the character table $\chi$ is such that $\chi$ is constant on a union of conjugacy classes forming a subgroup (which is then normal) and on its cosets, then $\chi$ is lifted from $D(\chi)$ via that subgroup.

**Theorem 2.64.** Let $G$ be a finite group of order $g$ with center $C$ of order $c$. The degrees of the irreducible representations $\chi_1, \ldots, \chi_h$ of $G$ divide $g/c = [G : C]$.

**Proof.** Let $\rho : G \to \text{GL}(V)$ be an irreducible representation of degree $n$. If $s \in C$, then $\rho_s : V \to V$ commutes with the action of $G$ by definition of the center, thus $\rho_s = \lambda_s \text{id}_V$ by Schur’s Lemma. Note that $\rho|_C : C \to \text{GL}(V)$ is a representation, thus the function $s \mapsto \lambda_s$ is a morphism of groups $C \to \mathbb{C}^\ast$.

Let $m \geq 1$ be an integer, set $V^{\otimes m} \overset{\text{def}}{=} \underbrace{V \boxtimes \cdots \boxtimes V}_m$ and form the box product representation

$$\rho^{\boxtimes m} \overset{\text{def}}{=} \underbrace{\rho \boxtimes \cdots \boxtimes \rho}_m : G^m \to \text{GL}(V^{\otimes m})$$

This is an irreducible representation of $G^m$ by Theorem 2.57. The subgroup $H \trianglelefteq G^m$ of those $(s_1, \ldots, s_m) \in C^m$ satisfying $s_1 s_2 \cdots s_m = 1$ lies in the kernel of $\rho^{\boxtimes m}$, thus we can drop it $D(\rho^{\boxtimes m}) : G^m/H \to \text{GL}(V^{\otimes m})$. Since $h \overset{\text{def}}{=} \text{ord}(H) = c^{m-1}$ (because $(s_1, \ldots, s_m) \in H$ if and only if \[ s_m = s_{m-1}^{-1} \cdots s_1^{-1} \]), by Corollary 3.20, we see that

$$\text{deg } D(\rho^{\boxtimes m}) = \dim V^{\otimes m} = n^m \mid \text{ord}(G^m/H) = g^m/c^{m-1},$$

37
we see that \((g/cn)^m\) lies in the \(\mathbb{Z}\)-submodule of \(\mathbb{Q}\) generated by \(\frac{1}{c}\). This holds for all \(m \geq 1\), so if we write \((g/cn)^m = p/q\) where \(p, q \in \mathbb{Z}\) are coprime integers and \(q > 0\), this implies that \(q^m\) divides \(c\) for all \(m \geq 1\), e.g. \(q = 1\).

(Remark : according to [l], this proof is due to John T. Tate.)

2.8 Induced representations

Given a finite group \(G\) and a subgroup \(H \leq G\), one easily produces representations of \(H\) by composing the morphism of groups \(\rho : G \to \text{GL}(V)\) with the inclusion map \(i : H \to G\), so that \(\text{Res}_H^G(\rho) \overset{\text{def}}{=} \rho|_H : H \to \text{GL}(V)\). The goal of this section is to show that this construction can be reversed, namely given a representation \(\theta : H \to \text{GL}(W)\), one can induce a representation \(\text{Ind}_H^G(\theta) : G \to \text{GL}(V)\). We will illustrate this induction process by formulating it in many different ways and deduce its properties in subsequent sections.

**Definition 2.65.** Let \(\rho : G \to \text{GL}(V)\) be a representation of \(G\), \(H \leq G\) a subgroup and \(\text{Res}_H^G(\rho) : H \to \text{GL}(V)\) is its restriction to the subgroup \(H \leq G\), i.e. \(\text{Res}_H^G(\rho) \overset{\text{def}}{=} \rho|_H\). Let \(W\) be a subrepresentation of \(\text{Res}_H^G(\rho)\) denoted by \(\theta : H \to \text{GL}(W)\) (\(\theta\) is again defined by “restriction”, but not at the same place; it is the restriction of \(\rho_s \in \text{GL}(V)\) to \(W\) for each \(s \in H\), which is possible because we assume \(W\) is \(H\)-stable, although not necessarily \(G\)-stable). For \(s \in G\), the vector subspace \(\rho_s(W) \subseteq V\) only depends on the left coset in which \(s\) is, because if \(s \in G\), \(t \in H\), we have

\[
\rho_{st}(W) = \rho_s(\rho_t(W)) = \rho_s(\theta_t(W)) = \rho_s(W).
\]

For \(s = sH \in G/H\) a left coset in \(G/H\), we define \(W_s \overset{\text{def}}{=} \rho_s(W)\) (the last argument shows that this is well-defined, i.e. does not depend on the choice of representative \(s\) for \(s = sH\)). We have \(\sum_{s \in G/H} W_s \subseteq V\).

We say that \(\rho : G \to \text{GL}(V)\) is **induced** by \(\theta : H \to \text{GL}(W)\) if as vector spaces, we have

\[
V = \bigoplus_{s \in G/H} W_s.
\]

Equivalently, each \(x \in V\) can be written uniquely in the form

\[
x = \sum_{s \in G/H} x_s, \quad x_s \in W_s.
\]

Another reformulation goes as follows. A section of the projection map \(\pi_H : G \to G/H\) is a map \(\phi : G/H \to G\) such that \(\pi_H \circ \phi = \text{id}_{G/H}\); the data of such a map is equivalent to a set \(R\), called a **system of left coset representatives** of \(G/H\), such that each \(\sigma \in G/H\) satisfies \(\sigma = rH\) for a unique \(r \in R\). Then as vector spaces, \(\rho : G \to \text{GL}(V)\) is induced by \(\theta : H \to \text{GL}(W)\) if and only if

\[
V = \bigoplus_{r \in R} \rho_r(W).
\]

In particular, we note that when \(V\) is induced by \(W\), we have \(\dim V = |G : H| \dim W\). It is also clear that the notion of being induced is independent of the choice of system of left coset representatives \(R\) (since \(W_s = \rho_r(W)\) does not depend on the choice of the group element \(r\) representing \(\sigma \in G/H\)).

**Remark 2.66.** Suppose \(\rho : G \to \text{GL}(V)\) is induced by \(\theta : H \to \text{GL}(W)\) where \(W \leq V\) is an \(H\)-subrepresentation of \(V\). Set \(g \overset{\text{def}}{=} \text{ord}(G)\) and \(h \overset{\text{def}}{=} \text{ord}(H)\). Fix a section of the map \(\pi_H : G \to G/H\), that is, fix a set of left coset representatives \(R = \{r_1, \cdots, r_l\}\) of \(G/H\) where \(l = g/h\). Extend the morphism of groups \(\theta : H \to \text{GL}(W) \subseteq \text{End}_C(W)\) to all of \(G\) as a map of sets by \(\theta(s) = 0 \in \text{End}_C(W)\) if \(s \in G \setminus H\). If we let \(\{w_1, \cdots, w_n\}\) be a basis of \(W\), then

\[
(\rho_i w_\beta \mid 1 \leq i \leq l, 1 \leq \beta \leq n) = (\rho_{r_1}(w_1), \cdots, \rho_{r_1}(w_n), \rho_{r_2}(w_1), \cdots, \rho_{r_2}(w_n))
\]
is an ordered basis of $V$ and with respect to this ordered basis, in block form, we have

$$\rho_s = (\theta_{r_i^{-1}sr_j})_{ij} = \begin{bmatrix}
\theta_{r_1^{-1}sr_1} & \theta_{r_2^{-1}sr_2} & \cdots & \theta_{r_1^{-1}sr_r} \\
\theta_{r_2^{-1}sr_1} & \theta_{r_2^{-1}sr_2} & \cdots & \theta_{r_1^{-1}sr_r} \\
\vdots & \vdots & \ddots & \vdots \\
\theta_{r_1^{-1}sr_1} & \theta_{r_2^{-1}sr_2} & \cdots & \theta_{r_1^{-1}sr_r}
\end{bmatrix}.$$ 

To see this, first note that for each $j$ there is a unique $i$ for which $r_i^{-1}sr_j \in H$ (because $sr_j$ belongs to a unique coset $r_iH$). Because $s$ was arbitrary in the previous argument and $r_i^{-1}sr_j \in H$ implies $r_j^{-1}s^{-1}r_i = (r_i^{-1}sr_j)^{-1} \in H$, we see that for each $i$, there is also a unique $j$ satisfying $r_i^{-1}sr_j \in H$. This is reflected in the fact that the action of $G$ on $\bigoplus_{\sigma \in G/H} W_\sigma$ permutes the subspaces $W_\sigma$ according to the cosets $\sigma$.

For $1 \leq \beta \leq n$ and $1 \leq j \leq \ell$, set $\rho_j(\beta) \overset{\text{def}}{=} \rho_{r_j}(\beta)$. Let $1 \leq i \leq \ell$ be the unique integer satisfying $r_i^{-1}sr_j \in H$. Letting $(\Theta_{ij})_{\alpha\beta}$ denote the matrix form of $\theta_{r_i^{-1}sr_j}$ over the basis $\{w_1, \cdots, w_n\}$, so that for $1 \leq \beta \leq n$,

$$\theta_{r_i^{-1}sr_j}(\beta) = \sum_{\alpha=1}^{n} \Theta_{ij}^{\beta\alpha} w_\alpha,$$

we have

$$\rho_s(v_{j,\beta}) = \rho_{sr_j}(\beta)$$

$$= \rho_{r_i r_i^{-1}sr_j}(\beta)$$

$$= \rho_{r_i} (\theta_{r_i^{-1}sr_j}(\beta))$$

$$= \rho_{r_i} (\Theta_{ij}^{\beta\alpha} w_\alpha)$$

$$= \rho_{r_i} \left( \sum_{\alpha=1}^{n} \Theta_{ij}^{\beta\alpha} w_\alpha \right)$$

$$= \sum_{\alpha=1}^{n} \Theta_{ij}^{\beta\alpha} \rho_{r_i}(w_\alpha)$$

$$= \sum_{\alpha=1}^{n} \Theta_{ij}^{\beta\alpha} v_{i,\alpha}.$$

Extending the definition of $\Theta_{ij}^{\alpha\beta}$ to $\Theta_{ij}^{\alpha\beta} = 0$ if $r_i^{-1}sr_j \notin G \setminus H$, (so that $\Theta_{ij}^{\alpha\beta}$ is the matrix form of $\theta_{r_i^{-1}sr_j}$ for all $1 \leq i, j \leq \ell$ and $s \in G$), we can write

$$\rho_s(v_{j,\beta}) = \sum_{i=1}^{\ell} \sum_{\alpha=1}^{n} \Theta_{ij}^{\beta\alpha} v_{i,\alpha},$$

which gives the matrix form of $\rho_s$ over the basis $\{v_{i,\alpha} \mid 1 \leq i \leq \ell, 1 \leq \alpha \leq n\}$.

**Proposition 2.67.** Let $\rho : G \to \text{GL}(V)$ be a representation of the finite group $G$ induced by the representation $\theta : H \to \text{GL}(W)$ of the subgroup $H \leq G$. Let $\chi$ denote the character of $\theta$ and write $\text{Ind}_{G/H}^G(\chi)$ denote the character of $\rho$. Set $R$ to be a system of left coset representatives for $G/H$. For any $s \in G$, we have

$$\text{Ind}_{G/H}^G(\chi)(s) = \sum_{r \in R} \chi(r^{-1}sr) = \frac{1}{|H|} \sum_{r \in G} \chi(t^{-1}st).$$
Proof. The first equality is clear after taking the trace in the matrix form of \( \rho_s \) in Remark 2.66. Note that if \( t \in rH \), letting \( t_r \) be such that \( t = r t_r \), we have \( t^{-1} s t = t_r^{-1} r^{-1} s r t_r \in H \) if and only if \( r^{-1} s r \in H \), in which case \( \chi(t^{-1} s t) = \chi(t_r^{-1} r^{-1} s r t_r) = \chi(r^{-1} s r) \). In other words, the whole coset \( rH \) contributes to \( h \chi(r^{-1} s r) \) to the sum over all \( t \in G \), so grouping the elements \( t \in G \) in the second sum according to their cosets gives the second equality.

Example 2.68. Let \( V = \mathbb{C}[G] \) be the regular representation of \( G \). It has a basis \( \{ e_t \}_{t \in G} \). Consider a subgroup \( H \subseteq G \) and let \( W = \{ e_t e_r \}_{t \in H} \subseteq \mathbb{C}[G] \) be the chosen subrepresentation of \( \text{Res}_H^G(\rho) : H \to \text{GL}(\mathbb{C}[G]) \) (notice that it is \( H \)-stable). The representation \( \theta : H \to \text{GL}(W) \) is the regular representation of \( H \). One clearly sees that \( \mathbb{C}[G] \) is induced by \( W \cong \mathbb{C}[H] \); this is just saying that \( G \) is a disjoint union of its left cosets. In particular, the regular representation of \( G \) is always induced by the trivial representation of its trivial subgroup.

To be able to use induced representations to character tables, it is also important to consider the relationship between conjugacy classes and subgroups. The most important properties are summarized in the following proposition.

Proposition 2.69. Let \( G \) be a group and \( H \) a subgroup. The group \( G \) acts on itself via conjugation, namely for \( s, t \in G \), \( s \cdot t \overset{def}{=} \text{sts}^{-1} \) defines a left action of \( G \) on itself. We also see that \( H \) acts on \( G \) via conjugation, and therefore also acts on \( H \). (All the actions considered in this proposition will be defined via conjugation.)

The conjugacy classes of \( G \) are disjoint union of orbits of the action \( H \circ G \). If \( C^G \) is a conjugacy class of \( G \) and \( C^H = \bigcup_{i \in I} C^H_i \) where the \( C^H_i \) are orbits of \( H \circ G \), pick \( t \in C^G \) and let \( \{ s_i \}_{i \in I} \subseteq G \) be such that \( s_i \cdot t \in C^H_i \). The maps

\[
s_j s_i^{-1} (-) s_j s_i^{-1} : C^H_i \to C^H_j
\]

are bijections, so that two conjugacy classes of \( H \) within the same conjugacy class of \( G \) have the same cardinality.

In particular, if \( H \) is a normal subgroup of the finite group \( G \), it is a union of conjugacy classes of \( G \) by definition, and those \( G \)-conjugacy classes are a disjoint union of conjugacy classes of \( H \). Therefore, if \( s \in H \) and \( c^H \) (resp. \( c^G \)) denotes the size of the conjugacy class of \( s \) in \( H \) (resp. in \( G \)), then \( c^H \) divides \( c^G \) and \( c^H_s = c^G_t \) for all \( t \in c^G_s \).

Proof. Two elements conjugate in \( H \) are also conjugate in \( G \), hence for any conjugacy class \( C^H \) of \( H \), we have \( C^H \subseteq \text{Orb}_G(C^H) \). Write \( C^G \overset{def}{=} \text{Orb}_G(C^H) \) as a union of conjugacy classes of \( H \). Orbits of an action always partition the set being acted on (this set being \( C^G \) considered as an \( H \)-action), which gives the first claim. If \( C^H_i \overset{def}{=} \text{Orb}_H(t) \), then \( s \cdot (-) : C^H_i \to C^H_{s \cdot i} \) is bijective (with inverse \( s^{-1} \cdot (-) \)), which proves the second claim. The particular case where \( H \) is normal in \( G \) and \( G \) is finite follows since the second claim implies \( c^H_s = c^H_t \) for all \( t \in c^G_s \) and the first claim implies that \( c^G_s \) is partitioned by subsets of size \( c^H_s \), hence \( c^G_s \) is a multiple of \( c^H_s \).

Example 2.70. We now consider the computation of a character table; the one of the alternating group \( A_4 \). Recall that \( A_4 \) is a normal subgroup of \( S_4 \) since \( |S_4 : A_4| = 2 \) and any subgroup of a group which has index 2 is normal. Therefore, we can apply Proposition 2.69 and our knowledge of the conjugacy classes of \( S_4 \) to compute the conjugacy classes of \( A_4 \). Also note that by the Orbit-Stabilizer theorem, the size of the conjugacy classes have to divide the order of the group. 33 In Example 2.52, we have seen that the conjugacy classes of \( S_4 \) correspond to permutations of the same cycle structure. Write the elements of \( S_4 \) in cycle notation, e.g. \((124)\) is the permutation \( 1 \mapsto 2 \mapsto 4 \mapsto 1 \) and \( 3 \mapsto 3 \). In \( A_4 \), there are three types of cycle structures, namely

\[
(1)(2)(3)(4), \quad (12)(34), \quad (123).
\]

The identity is obviously in its own conjugacy class. As for the cycle structure \((12)(34)\), there are three such permutations, so the orbits in this cycle structure have order 1 or 3; since their cycle structure is left
invariant by the action of $A_4$ (for example, conjugation of $(12)(34)$ by $(123)$ gives $(23)(14)$), this is an orbit of $A_4$ of order 3. For the remaining eight 3-cycles, we can split them down in two rows, namely

$$
(123) \quad (142) \quad (134) \quad (243) \\
(132) \quad (124) \quad (143) \quad (234)
$$

Note that if $\sigma$ is the permutation in the first row, then $\sigma^2$ is below it in the second row. Since the $S_4$-orbit corresponding to this cycle structure has order 8, the $A_4$-orbit of such permutations has to divide 8 and 12, thus has to divide 4. One readily sees that all these permutations have a fixed point and conjugation by elements of $A_4$ moves this fixed point over all of $\{1, 2, 3, 4\}$, hence an orbit contains at least four distinct elements, and therefore has order 4. One can easily see that the permutations listed in the top row are all conjugate: letting $\cdot$ denote action via conjugation,

$$(12)(34) \cdot (123) = (214) = (142), \quad (13)(24) \cdot (123) = (341) = (134), \quad (14)(23) \cdot (123) = (432) = (243),$$

so that the top row is a conjugacy class of $A_4$, and thus so does the bottom row.

Noticing that $K \overset{\text{def}}{=} \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ is a subgroup which is a union of conjugacy classes of $A_4$, we see that it is a normal subgroup of $A_4$. Since $A_4/K \cong \mathbb{Z}/3\mathbb{Z}$ is abelian, all its irreducible representations have degree 1, thus lifting them to $A_4$ gives us three irreducible characters of $A_4$ (namely $\chi_1, \chi_2$ and $\chi_3$ in the table below; since $\mathbb{Z}/3\mathbb{Z}$ is cyclic, a morphism of groups $\mathbb{Z}/3\mathbb{Z} \to \mathbb{C}^\times$ is the same as a choice of a root of the equation $x^3 = 1$, hence the formula for the three characters. They are given below:

write $\omega = e^{2\pi i/3} = -\frac{1+\sqrt{3}i}{2}$, so that

$$
\begin{array}{cccc}
   c_s & 1 & 3 & 4 & 4 \\
   A_4 & 1 & x & t & t^2 \\
   \chi_1, \text{ trivial} & 1 & 1 & 1 & 1 \\
   \chi_2 & 1 & 1 & \omega & \omega^2 \\
   \chi_3 & 1 & 1 & \omega^2 & \omega \\
   \psi & 3 & -1 & 0 & 0
\end{array}
$$

where the last row $\psi$ was filled using Corollary 2.27 and the column orthogonality relations (cf. Theorem 2.34). Another way to obtain $\psi$ is to notice that $A_4$ acts doubly transitively on $\{1, 2, 3, 4\}$. To prove this, recall that $S_4$ does, and if $S_4$ uses an odd permutation $\sigma$ to map $\alpha$ to $a$ and $\beta$ to $b$, write $\{1, 2, 3, 4\} = \{a, b, c, d\}$ so that the permutation $(cd)\sigma$ also does the trick and is an even permutation. The corresponding standard representation with character $\psi = \chi - \chi_1$ is irreducible by Theorem 2.46.

**Example 2.71.** We compute a second character table, the one of the dihedral group

$$D_n = \langle r, s \mid r^n = s^2 = 1, sr^{-1}s^{-1} = r^{-1} \rangle$$

where $n \geq 2$. Because there are many possible conjugacy classes, we list the character values as a function of the elements instead of the conjugacy classes. In other words, any element can be written in the form $r^k$ or $sr^k$ for $0 \leq k \leq n-1$ and we give the character as a function of $k$. Note that the number of conjugacy classes will be deduced from the number of irreducible characters.

The subgroup $\langle r \rangle$ is abelian and has index 2 (thus is normal), hence every irreducible representation of $D_n$ has degree at most 2 by Corollary 2.54. We begin with the irreducible characters of degree 1. It suffices to compute what are the possible group homomorphisms $\rho : D_n \to \mathbb{C}^\times$. It suffices to know the image of $\rho_r$ and $\rho_s$; they must satisfy the relations

$$\rho^2_s = \rho^n_s = 1, \quad \rho_s\rho_r \rho_s^{-1} = \rho_{r^{-1}} \quad \implies \quad \rho_r = \rho_{r^{-1}}^{-1} \quad \implies \quad \rho^2_r = 1.$$
It follows that \( \rho_r, \rho_s \in \{-1, 1\} \). Furthermore, if \( n \) is odd, \( \rho_r^n = \rho_s^n = 1 \) implies \( \rho_r = 1 \). We must split into two cases:

<table>
<thead>
<tr>
<th>( D_n )</th>
<th>( r^k )</th>
<th>( sr^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi_1 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_2 )</td>
<td>1</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \psi_3 )</td>
<td>((-1)^k)</td>
<td>((-1)^k)</td>
</tr>
<tr>
<td>( \psi_4 )</td>
<td>((-1)^k)</td>
<td>((-1)^k+1)</td>
</tr>
</tbody>
</table>

Note that our choice of notation is such that \( \psi_1 \) and \( \psi_2 \) are given by the same formulas whether \( n \) is even or not. We now turn to degree 2 representations. To proceed, we use the normal subgroup \( \langle r \rangle \) and induce its irreducible representations to \( D_n \). Since \( \langle r \rangle \simeq \mathbb{Z}/n\mathbb{Z} \), an irreducible character of \( \mathbb{Z}/n\mathbb{Z} \) is equivalent to a choice of an \( n \)th root of unity. For each \( 0 \leq j \leq n - 1 \), let \( \omega_j \defeq e^{2\pi ij/n} \) and let \( \rho^j : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^* \) be the corresponding character, so that

\[
\text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{D_n}(\rho^j)_r = \begin{bmatrix} \omega_j^k & 0 \\ 0 & \omega_j^{-k} \end{bmatrix}, \quad \text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{D_n}(\rho^j)_s = \begin{bmatrix} 0 & \omega_j^{-k} \\ \omega_j^k & 0 \end{bmatrix}.
\]

To obtain these formulas via computations, use the formula given in Remark 2.66. Their characters are easily computed since we have them in matrix form. Let \( \chi_j \) be the character associated to \( \text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{D_n}(\rho^j) \), so that

\[
\chi_j(r^k) = \omega_j^k + \omega_j^{-k}, \quad \chi_j(sr^k) = 0.
\]

Note that it is not true in general that representations induced by an irreducible representation are irreducible (c.f. Example 2.68 for the example of the regular representation), hence we still have to check that \( \chi_j \) is irreducible:

\[
(\chi_j, \chi_j) = \frac{1}{2n} \sum_{k=0}^{n-1} \chi_j(r^k)^2 = \frac{1}{2n} \sum_{k=0}^{n-1} (\omega_j^{2k} + \omega_j^{-2k} + 2) = 1 + \frac{1}{n} \sum_{k=0}^{n-1} \omega_j^{2k}.
\]

To understand the last equality, the substitution \( k \mapsto n - k \) gives

\[
\sum_{k=0}^{n-1} \omega_j^{-2k} = \sum_{k=n}^{n-1} \omega_j^{-2(n-k)} = \sum_{k=1}^{n} \omega_j^{2k},
\]

and since \( \omega_j^{2n} = 1 \), we can replace the term with \( k = n \) by \( k = 0 \) so that

\[
\sum_{k=0}^{n-1} \omega_j^{-2k} = \sum_{k=0}^{n-1} \omega_j^{-2k} = \sum_{k=0}^{n-1} \omega_1^{2jk}.
\]

To check if \( \chi_j \) is irreducible, it suffices to determine if the latter sum equals zero. Note that \( \chi_j = \chi_{n-j} \), so we can restrict our attention to \( 0 \leq j \leq n/2 \). We can even remove the case \( j = 0 \) (since \( \chi_0 = \psi_1 + \psi_2 \)) and \( j = n/2 \) when \( n \) is even (since \( \chi_{n/2} = \psi_3 + \psi_4 \)), so in fact, we restrict our attention to \( 1 \leq j < n/2 \). Irreducibility can then be checked in two different method. The first method consists in computing the above sum explicitly: since \( 1 \leq j < n/2 \), \( \omega_1^{2j} \neq 1 \), so we set \( x = \omega_1^{2j} \) in the geometric progression formula

\[
(x - 1) \left( \sum_{k=0}^{n-1} x^k \right) = x^n - 1 \quad \Rightarrow \quad \left( \sum_{k=0}^{n-1} \omega_1^{2jk} \right) = \frac{\omega_1^{2jn} - 1}{\omega_1^{2j} - 1} = 0.
\]

The second method is by showing directly that no one-dimensional subspace of \( \mathbb{C}^2 \) is \( D_n \)-stable under the action of \( \text{Ind}_{\mathbb{Z}/n\mathbb{Z}}^{D_n}(\rho^j) \). To see this, if \( v \in \mathbb{C}^2 \setminus \{0\} \), the orbit of \( v \) via the action of the subgroup \( \langle r \rangle \leq D_n \) spans a line if and only if \( v \) lies on the coordinate axes of \( \mathbb{C}^2 \), but since \( s \in D_n \) permutes the axes, the orbit of \( v \) has to span \( \mathbb{C}^2 \).

To show that there are no more irreducible characters of \( D_n \), we split into two cases:
- Assume \( n \) is even, so that there are \( \frac{n}{2} - 1 \) values of \( j \) satisfying \( 1 \leq j < n/2 \). Since

\[
2n = \operatorname{ord}(D_n) = 4 \cdot 1^2 + ((n/2) - 1) \cdot 2^2,
\]

we have found all the irreducible representations of \( D_n \), hence the following character table:

<table>
<thead>
<tr>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
<th>( \psi_3 )</th>
<th>( \psi_4 )</th>
<th>( \chi_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_n )</td>
<td>( \tau^k )</td>
<td>( \tau^k )</td>
<td>( \tau^k )</td>
<td>( \omega_j^k + \omega_j^{-k} )</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_2 )</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>( \psi_3 )</td>
<td>(-1)^k</td>
<td>(-1)^k</td>
<td>(-1)^{k-1}</td>
<td></td>
</tr>
<tr>
<td>( \psi_4 )</td>
<td>(-1)^k</td>
<td>(-1)^k</td>
<td>(-1)^{k-1}</td>
<td></td>
</tr>
</tbody>
</table>

In particular, there are \( n/2 - 1 + 4 = n/2 + 3 \) irreducible characters/conjugacy classes in \( D_n \) for \( n \) even.

- Assume \( n \) is odd, so that there are \( \lfloor n/2 \rfloor = \frac{n-1}{2} \) values of \( j \) satisfying \( 1 \leq j < n/2 \). Since

\[
2n = \operatorname{ord}(D_n) = 2 \cdot 1^2 + \frac{n-1}{2} \cdot 2^2,
\]

we have all the irreducible characters of \( D_n \), hence

<table>
<thead>
<tr>
<th>( \psi_1 )</th>
<th>( \psi_2 )</th>
<th>( \chi_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_n )</td>
<td>( \tau^k )</td>
<td>( \tau^k )</td>
</tr>
<tr>
<td>( \psi_1 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \psi_2 )</td>
<td>1</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

It follows that there are \( \frac{n-1}{2} + 2 = \frac{n+3}{2} \) irreducible characters/conjugacy classes in \( D_n \) for \( n \) odd.

**Example 2.72.** Consider the alternating group \( G = A_5 \). We already know of one irreducible character of \( H \), namely the trivial character which we denote by \( \chi_1 \). The doubly transitive action \( S_5 \cap \{1, 2, 3, 4, 5\} \) is also doubly transitive when we restrict it to the subgroup \( A_5 \); to see this, if \( a, b, c, d \) denote distinct integers, the following table shows how to find a permutation in \( A_5 \) which maps a pair of distinct integers to another:

\[
\begin{align*}
[\begin{array}{cc}
a & b \\
c & d
\end{array}] & \longmapsto (abcd) \\
[\begin{array}{cc}
a & a \\
c & d
\end{array}] & \longmapsto (ab)(cd)
\end{align*}
\]

\[
\begin{align*}
[\begin{array}{cc}
a & a \\
c & c
\end{array}] & \longmapsto (ac)(bd) \\
[\begin{array}{cc}
a & b \\
c & a
\end{array}] & \longmapsto (abc).
\end{align*}
\]

This gives a second irreducible character by counting fixed points of the action (c.f. Theorem 2.46); we denote this character by \( \chi_2 \) and note that it has degree 4. Recall that conjugacy classes in \( S_5 \) are characterized by the possible cycle structures (c.f. Example 2.52), hence the conjugacy classes of \( A_5 \) of a given cycle structure partition the set of even permutations of that cycle structure. In other words, we have at least 4 conjugacy classes, namely those corresponding to the even cycle structures \( 1 \), \( (123) \), \( (12)(34) \) and \( (12345) \). We now check that only the latter cycle structure splits in two conjugacy classes, namely that of \( (12345) \) and of \( (12354) \).

Let \( a, b, c, d, e \in \{1, 2, 3, 4, 5\} \) be distinct integers and \( \sigma \in S_5 \) be a permutation such that \( \sigma(123)\sigma^{-1} = (abc) \). If \( \sigma \) is even, \( (123) \) and \( (abc) \) are conjugate, and if \( \sigma \) is odd, then \( (de)\sigma(123)\sigma^{-1}(de)^{-1} = (de)(abc)(de) = (abc) \) where \( (de)\sigma \) is even, so the 3-cycles form a conjugacy class in \( A_5 \). If \( \sigma \in S_5 \) is such that \( \sigma(12)(34)\sigma^{-1} = (ab)(cd) \) and \( \sigma \) is even, \( (12)(34) \) and \( (ab)(cd) \) are conjugate; if it is odd, then
\(\sigma(12)\) sends \((12)(34)\) to \((ab)(cd)\) via conjugation. Finally, if \(\sigma(12345)\sigma^{-1} = (abcde)\) and \(\sigma\) is odd, then 
\(\sigma(12345)(45)^{-1}\sigma^{-1} = \sigma(12354)\sigma^{-1} = (abde)\), so it remains to see that \((12345)\) and \((12354)\) are not conjugate. For \(\sigma \in S_5\) and \(a \in \{1, 2, 3, 4, 5\}\), write \(\sigma_a \defeq \sigma(a)\). If \(\sigma(12345)\sigma^{-1} = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5) = (12354)\) and \(\sigma(1) = a\), then

\[
(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \in \{(1, 2, 3, 5, 4), (2, 3, 5, 4, 1), (3, 5, 4, 1, 2), (5, 4, 1, 2, 3), (4, 1, 2, 3, 5)\}
\]

and one can check that the corresponding values of \(\sigma\) are the odd permutations

\[
\{(45), (1235), (134)(25), (153)(24), (3214)\}.
\]

It follows that we have five conjugacy classes in \(A_5\), and therefore we are missing three irreducible characters \(\chi_3, \chi_4, \chi_5\). Their degrees \(n_3, n_4, n_5\) have to satisfy the equation

\[
1^2 + 4^2 + n_3^2 + n_4^2 + n_5^2 = |A_5| = 60,
\]

\(\text{e.g. } n_3^2 + n_4^2 + n_5^2 = 43\). We cannot have \(n_i \geq 6\) since \(7^2 > 43\) and the equation \(a^2 + b^2 = 43 - 6^2 = 7\) has no solution in the integers. Therefore \(1 \leq n_3, n_4, n_5 \leq 5\). There can be at most one of the \(n_i\)'s equal to 5 (since \(50 > 43\)) and at least one, since if \(n_3, n_4, n_5 \leq 4\), they cannot all be equal to 4 \((48 \neq 43\)) and if only one of them is smaller than 4, the resulting sum is either equal to 45 or is less than 40. Therefore we can set \(n_5 = 5\). The equation \(n_3^2 + n_4^2 = 43 - 25 = 18\) has a unique solution in positive integers, namely \(n_3 = n_4 = 3\).

Using elementary combinatorics, one determines the size of each conjugacy class. We summarize what we have up to now in the following table, to which we temporarily adjoin the symmetric and alternating powers of \(\chi_2\) (which are \textbf{not} irreducible) to continue filling in the table (c.f. Proposition 2.12): 

<table>
<thead>
<tr>
<th>(c_s)</th>
<th>(1)</th>
<th>(20)</th>
<th>(15)</th>
<th>(12)</th>
<th>(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_5)</td>
<td>1</td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_2)</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(\chi_3)</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi_4)</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi_5)</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\chi_2^\sigma)</td>
<td>10</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\chi_2^\alpha)</td>
<td>6</td>
<td>0</td>
<td>-2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Computing explicitly the square of a permutation is not necessary to evaluate these characters via the formulas of Proposition 2.12 since the conjugacy classes of the first three classes squared are trivially evaluated and conjugacy class of a 5-cycle squared is again that of a 5-cycle, and \(\chi_2\) takes equal values on both conjugacy classes. Note that an element of \(A_5\) is conjugate to its inverse ; this is trivial for the first three conjugacy classes, and we note that

\[
(12345)^{-1} = (54321) = (15432) = (25)(34)(12345)((25)(34))^{-1}.
\]

Therefore, we can compute \(\langle \chi_i, \chi_j \rangle_{A_5}\) by summing the products of each of the rows \(c_s, \chi_i\) and \(\chi_j\) over all columns and then dividing by \(|A_5| = 60\). We deduce

\[
\langle \chi_2^\sigma, \chi_1 \rangle = 1, \quad \langle \chi_2^\sigma, \chi_2 \rangle = 1, \quad \langle \chi_2^\sigma - \chi_1 - \chi_2, \chi_2^\sigma - \chi_1 - \chi_2 \rangle = 1
\]

which implies that \(\chi_5 = \chi_2^\sigma - \chi_1 - \chi_2\). Similarly, we can compute

\[
\langle \chi_2^\alpha, \chi_1 \rangle = \langle \chi_2^\alpha, \chi_2 \rangle = \langle \chi_2^\alpha, \chi_5 \rangle = 0, \quad \langle \chi_2^\alpha, \chi_2^\alpha \rangle = 2,
\]

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which implies that $\chi_{2,2}^2 = \chi_3 + \chi_4$. To see this, write $\chi_{2,2}^2 = a\chi_3 + b\chi_4$ where $a, b$ are non-negative integers. Then $\langle \chi_{2,2}^2, \chi_{2,2}^2 \rangle = a^2 + b^2 = 2$ implies $a = b = 1$. We now claim that we have gathered enough representation-theoretic information to use the orthogonality relations and find the remaining two characters. We introduce variables to compute them:

$$c_s = \begin{pmatrix} 1 & 20 & 15 & 12 & 12 \\ A_5 & (123) & (12)(34) & (12345) & (12354) \\ \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 4 & 1 & 0 & -1 & -1 \\ \chi_3 & 3 & x_1 & x_2 & x_3 & x_4 \\ \chi_4 & 3 & y_1 & y_2 & y_3 & y_4 \\ \chi_5 & 5 & -1 & 1 & 0 & 0 \end{pmatrix}$$

From the relation $\chi_{2,2}^2 = \chi_3 + \chi_4$, we deduce

$$x_1 + y_1 = 0, \quad x_2 + y_2 = -2, \quad x_3 + y_3 = 1, \quad x_4 + y_4 = 1.$$  

From considering the products $\langle \chi_i, \chi_3 \rangle$ and $\langle \chi_i, \chi_4 \rangle$ for $i = 1, 2, 5$, we get the matrix of linear relations

$$\begin{pmatrix} 20 & 15 & 12 \\ 20 & 0 & -12 \\ -20 & 15 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -15 \\ -12 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 + x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 + y_4 \end{pmatrix}$$

Since the column vectors associated to the pairs $((12)(34), (12345))$ and $((12)(34), (12354))$ are orthogonal, we deduce $x_3 + y_3 = 1 = x_4 + y_4$, and in particular, $x_3 = y_4$ and $x_4 = y_3$. So it remains to determine $x_3$, and for this we use the orthogonality of the two last columns:

$$2 + 2x_3(1 - x_3) = 1 + (-1)^2 + x_3 x_4 + y_3 y_4 = 0 \implies x_3^2 - x_3 - 1 = 0,$$

so it follows that $x_3 \in \left\{ \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right\}$. Letting $x_3 = y_4$ be one of the two possible values and using the row orthogonality relations, we see that $y_3$ takes the other value, so it does not matter which root of this equation we take since both solutions are actually necessary. We finally display the completed character table:

$$c_s = \begin{pmatrix} 1 & 20 & 15 & 12 & 12 \\ A_5 & (123) & (12)(34) & (12345) & (12354) \\ \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 4 & 1 & 0 & -1 & -1 \\ \chi_3 & 3 & 0 & -1 & \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\ \chi_4 & 3 & 0 & -1 & \frac{1 - \sqrt{5}}{2} & \frac{1 + \sqrt{5}}{2} \\ \chi_5 & 5 & -1 & 1 & 0 & 0 \end{pmatrix}$$

Another technique that could have been used instead of linear algebra is to notice that conjugation by $\phi \in \text{Aut}(S_5)$ which permutesthe only the two last conjugacy classes. The characters $\chi_3$ and $\chi_4$ cannot be equal on the conjugacy classes $(12345)$ and $(12354)$, otherwise the corresponding columns would be equal and thus not orthogonal. It follows that $\chi_3^\phi = \chi_4$, hence

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_4, \quad x_4 = y_3$$

and the column orthogonality relations allow computing the remaining coefficients directly.
Chapter 3

Representations as $\mathbb{C}[G]$-modules

In this section, we will start assuming that the reader might be familiar with some notions of category theory; if the reader isn’t, ignoring those notions is fine since we will not use them explicitly yet (perhaps in subsequent chapters). The complex group algebra of a finite group will prove very useful to the understanding of representations and we dedicate this chapter to this notion. Representations will be seen as modules over the group algebra, thus allowing the theory of left modules over a non-commutative ring to apply to representations.

We still consider exclusively finite-dimensional vector spaces and all groups denoted by $G$, $H$, $K$ denote finite groups unless mentioned otherwise. We apologize for the ambiguity that context should make obvious what that letter means, and if things seem confusing, precisions will be added.

3.1 Generalities

Definition 3.1. Let $G$ be a finite group and $k$ a non-commutative unital ring. The group algebra of $G$ over $k$, denoted by $k[G]$, is constructed as follows. As a set, $k[G] \overset{def}{=} \text{Hom}_{\text{Set}}(G, k)$ is the set of all functions $a : G \to k$. It is clearly a non-commutative ring with operations given by pointwise addition and multiplication. It is also a free left $k$-module with $k$-basis $\{e_s \}_{s \in G}$, so that its elements $a \in k[G]$ can be uniquely written in the form $a = \sum_{s \in G} a_s e_s$ where $a_s = a(s)$. We take the convention of writing the basis elements of $k[G]$ as the group elements themselves, i.e. $a = \sum_{s \in G} a_s s$. Under this notation, the group is naturally a subset of its group algebra, i.e. $G \subseteq k[G]$.

If $V$ is a left $k[G]$-module, the subset $G \subseteq k[G]$ can act on vectors $v \in V$ via $s \cdot v \overset{def}{=} e_s v$. This induces a $k$-linear left group action of $G$ on $V$, i.e. a morphism of groups $G \to \text{GL}_k(V)$ where $\text{GL}_k(V)$ denotes the group of $k$-linear automorphisms of $V$ (when $k = \mathbb{C}$, this is a complex representation of $G$).

If $V, W$ are left $k[G]$-modules, we can take their tensor product as $k$-modules to form $V \otimes_k W$ (c.f. Remark 1.16), which is then also a left $k[G]$-module via $s \cdot (v \otimes w) \overset{def}{=} (s \cdot v) \otimes (s \cdot w)$. We also give $\text{Hom}_k(V, W)$ a left $k[G]$-module structure via $(s \cdot f)(v) \overset{def}{=} s \cdot f(s^{-1} \cdot v)$.

We will constantly work with finitely generated left $k[G]$-modules, hence we call such an object a $\mathbb{C}[G]$-module and denote the abelian category of (finitely generated left) $k[G]$-modules by $k[G] \text{-Mod}_{\text{fin}}$. When $k$ is left-Noetherian, so is $k[G]$, hence this category is abelian (just as the category of finitely generated $R$-modules is when $R$ is a left-Noetherian non-commutative ring), so left $k[G]$-modules inherit the notion of kernel, cokernel, image and biproducts (which we will call direct sums). Furthermore, the tensor product of left $k[G]$-modules defined above makes $k[G] \text{-Mod}_{\text{fin}}$ into a symmetric monoidal category.
Remark 3.2. Let $G$ be a finite group and $k$ be a non-commutative unital ring. If $f_1 = \sum_{s \in G} a_s s$ and $f_2 = \sum_{s \in G} b_s s$ are in $k[G]$, then

$$f_1 + f_2 = \sum_{s \in G} (a_s + b_s) s, \quad f_1 f_2 = \sum_{s \in G} \left( \sum_{t u = s, t, u \in G} a_t b_u \right) s = \sum_{s \in G} \left( \sum_{t \in G} a_{st^{-1}} b_t \right) s.$$

This turns $k[G]$ into a unital ring \((1_{k[G]} \overset{\text{def}}{=} 1_k 1_G)\) which is commutative if and only if $k$ is commutative and $G$ is abelian. One can also see that multiplication can be interpreted as follows: for $a_s, b_t \in k$ and $s, t \in G$, define \((a_s b_t) \overset{\text{def}}{=} (a_s b_t)(st)\), where $s$ and $t$ are multiplied in $G$ and $a_s$ and $b_t$ are multiplied in $k$; extend this definition by $k$-bilinearity to obtain the general definition.

Proposition 3.3. Let $G$ be a finite group. There is a canonical bijection

$$\{ \rho : G \to \text{GL}(V) \mid \rho \text{ is a complex representation } \} \xrightarrow{1:1} \{ V \text{ a finitely generated } \mathbb{C}[G]\text{-module} \}.$$

The \(\mathbb{C}[G]\)-submodules of $V$ are in one-to-one correspondence with $G$-stable complex vector subspaces $W \subseteq V$. More precisely, we have an isomorphism of abelian symmetric monoidal categories

$$\text{Rep}_\mathbb{C}(G) \simeq \mathbb{C}[G]\text{-Mod}_\text{fin}.$$

In other words, this isomorphism preserves kernels, cokernels, images, quotients, direct sums and tensor products. It also respects the notion of “hom” defined in both categories (i.e. the Hom\(_\mathbb{C}\)(-,-) of two representations and the one defined in Definition 3.1).

**Proof.** Given a representation \(\rho : G \to \text{GL}(V)\), we can turn the vector space $V$ into a \(\mathbb{C}[G]\)-module by letting $s \cdot v \overset{\text{def}}{=} \rho_s(v)$ and extend this definition to \(\mathbb{C}[G]\) by \(\mathbb{C}\)-linearity. This defines a \(\mathbb{C}[G]\)-module structure on $V$. Conversely, if $V$ is a \(\mathbb{C}[G]\)-module, then \(\rho : G \to \text{GL}(V)\) defined by \(\rho_s(v) \overset{\text{def}}{=} s \cdot v\) is a morphism of groups. These correspondences are inverse to each other, thus the first part of the statement.

As for the isomorphism of categories, it is clear that this definition respects morphisms of representations / \(\mathbb{C}[G]\)-modules since “it does nothing on the maps”, i.e. a $G$-linear map $f : V \to W$ gets mapped to exactly the same map of vector spaces $f : V \to W$, except that it is considered as a \(\mathbb{C}[G]\)-linear map in \(\mathbb{C}[G]\)-Mod\(_\text{fin}\). The notions of kernel, cokernel, image, quotient and direct sum agree with the underlying notions applied to vector spaces, so we have nothing to prove there. As for the tensor product, one sees that the definitions agree on simple tensors, completing the proof. The same argument applies to the construction $(V, W) \mapsto \text{Hom}_\mathbb{C}(V, W)$ in both categories.

Definition 3.4. Let $G$ be a finite group. A \(\mathbb{C}[G]\)-module $V$ is called irreducible or simple if the corresponding representation is irreducible; in other words, $V$ is irreducible if it is a simple \(\mathbb{C}[G]\)-module, i.e. its set of \(\mathbb{C}[G]\)-submodules has precisely two elements, 0 and $V$ (so that $V \neq 0$). It is called completely reducible if it can be written as a direct sum of irreducible \(\mathbb{C}[G]\)-submodules. The character of a \(\mathbb{C}[G]\)-module is the character of the corresponding representation.

Example 3.5. Under the correspondence of Proposition 3.3, note that the \(\mathbb{C}[G]\)-module \(\mathbb{C}[G]\) corresponds to the left regular representation of $G$, which we fortunately also denoted by \(\mathbb{C}[G]\). Therefore, we have a decomposition of \(\mathbb{C}[G]\) as a direct sum of irreducible \(\mathbb{C}[G]\)-modules.

Theorem 3.6. Every exact sequence of \(\mathbb{C}[G]\)-modules splits, or equivalently, for every \(\mathbb{C}[G]\)-module $V$ and \(\mathbb{C}[G]\)-submodule $W \subseteq V$, there exists a \(\mathbb{C}[G]\)-submodule $W_0$ of $V$ such that $V = W \oplus W_0$ as \(\mathbb{C}[G]\)-modules.
Remark 3.7. In light of Proposition 3.3, we now think of the terms “linear representation” and \( \mathbb{C}[G] \)-module” as being equivalent. Since we assume that all our \( \mathbb{C}[G] \)-modules are finitely generated, our vector spaces over \( \mathbb{C} \) corresponding to them are finite-dimensional.

Theorem 3.8. (Artin-Wedderburn theorem) Let \( G \) be a finite group.

(i) Every \( \mathbb{C}[G] \)-module is completely reducible, i.e. can be written as a direct sum of irreducible \( \mathbb{C}[G] \)-submodules.

(ii) Every \( \mathbb{C}[G] \)-module is projective and injective, i.e. every exact sequence of \( \mathbb{C}[G] \)-modules is split.

(iii) Let \( \rho^l : G \to GL(\mathbb{C}^n_1), \cdots, \rho^h : G \to GL(\mathbb{C}^n_h) \) be the \( h \) irreducible representations of \( G \) written in some matrix form. Since this turns \( \mathbb{C}^n_i \) into a \( \mathbb{C}[G] \)-module, let

\[
\tilde{\rho}^l : \mathbb{C}[G] \to \text{Mat}_{n_1 \times n_1}(\mathbb{C}), \cdots, \tilde{\rho}^h : \mathbb{C}[G] \to \text{Mat}_{n_h \times n_h}(\mathbb{C})
\]

be the corresponding morphism of \( \mathbb{C} \)-algebras defined by

\[
\tilde{\rho}^l \left( \sum_{s \in G} a_s s \right) \overset{def}{=} \sum_{s \in G} a_s \rho^l_s.
\]

Taking the product map for \( i = 1, \cdots, h \), we obtain a map

\[
\tilde{\rho} = (\tilde{\rho}^1, \cdots, \tilde{\rho}^h) : \mathbb{C}[G] \xrightarrow{\simeq} \prod_{i=1}^h \text{Mat}_{n_i \times n_i}(\mathbb{C})
\]

where \( n_1, \cdots, n_h \) are the degrees of the irreducible representations of \( G \) (i.e. the dimensions of the simple \( \mathbb{C}[G] \)-modules).

In other words, the \( \mathbb{C} \)-algebra \( \mathbb{C}[G] \) is semisimple (this means one of the equivalent notions of (i), (ii) or (iii) ; that they are equivalent is precisely Artin-Wedderburn’s theorem).

Proof. By Maschke’s theorem, if \( V \) is a non-irreducible \( \mathbb{C}[G] \)-module, we can write \( V = W \oplus W' \) where \( W, W' \subseteq V \) are two \( \mathbb{C}[G] \)-submodules ; the result in (i) then follows by induction on \( \dim_{\mathbb{C}} V \). Since every irreducible representation is a direct summand of \( \mathbb{C}[G] \), if \( V \) is a \( \mathbb{C}[G] \)-module and \( (W_1, \cdots, W_h) \) are irreducible \( \mathbb{C}[G] \)-submodules such that \( V \simeq \bigoplus_{i=1}^h W_i \), then \( V \) is isomorphic to a direct summand of \( \mathbb{C}[G]^\oplus h \), hence is projective. It follows that every exact sequence of \( \mathbb{C}[G] \)-modules splits, thus every \( \mathbb{C}[G] \)-module is also injective, which proves (ii).

Onto (iii). We first show that \( \tilde{\rho} \) is injective. Suppose that \( \sum_{s \in G} a_s s \) maps to zero, i.e. for all irreducible representations of \( G \), we have \( \sum_{s \in G} a_s \rho^l_s = 0 \). We show that \( a_1 = 0 \). Notice that for each \( t \in G \), we have

\[
0 = \rho^l_t \left( \sum_{s \in G} a_s \rho^l_s \right) \rho^l_s = \sum_{s \in G} a_s \rho^l_{ts^{-1}} = \sum_{s \in G} a_{t^{-1}st} \rho^l_s
\]

hence by averaging over \( t \in G \) and taking the trace, we get \( \sum_{s \in G} \left( \frac{1}{g} \sum_{t \in G} a_{t^{-1}st} \right) \chi_i(s) = 0 \). Now \( \psi(s) \overset{def}{=} \frac{1}{g} \sum_{t \in G} a_{t^{-1}st} \) is a class function, and

\[
(\psi, \chi_i) = \frac{1}{g} \sum_{s \in G} \psi(s) \chi_i(s) = \frac{1}{g} \sum_{s \in G} \psi(s) \chi(s^{-1}) = \frac{1}{g} \sum_{s \in G} \psi(s^{-1}) \chi(s) = \frac{1}{g} \sum_{s,t \in G} a_{t^{-1}st} \chi_i(s) = 0.
\]
This means $\psi = 0$, and in particular $a_1 = \psi(1) = 0$, as desired. Let $t \in G$ and define $b_s \overset{\text{def}}{=} a_{st}$. Then since $\sum_{s \in G} a_s s$ maps to zero, the sum $\sum_{s \in G} b_s s$ maps to
\[
\sum_{s \in G} b_s \rho_s^i = \sum_{s \in G} a_{st} \rho_s^i = \sum_{s \in G} a_s \rho_s^i = \left( \sum_{s \in G} a_s \rho_s^i \right) \rho_t^i = 0.
\]
Thus $b_1 = a_t = 0$, so that $\tilde{\rho}$ is injective. On the other hand,
\[
\dim_\mathbb{C} \mathbb{C}[G] = g = \sum_{i=1}^h n_i^2 = \dim_\mathbb{C} \prod_{i=1}^h M_{n_i}(\mathbb{C}),
\]
so $\tilde{\rho}$ is bijective. It is clearly a map of $\mathbb{C}$-algebras, thus an isomorphism of $\mathbb{C}$-algebras.

**Remark 3.9.** Alternatively, the proof of (iii) could go as follows: assume $\tilde{\rho}$ is not surjective, so that there exists a non-zero linear form $\ell : \prod_{i=1}^h M_{n_i}(\mathbb{C}) \to \mathbb{C}$ which vanishes on $\im \tilde{\rho}$. Let $E_{ij}^k \in \prod_{i=1}^h M_{n_i}(\mathbb{C})$ be the zero matrix in all components $k' \neq k$ of this product, and be the matrix $E_{ij}$ in the $k$th component (i.e., has a 1 in the $(i, j)$th component of the matrix of size $n_i$). Then by Remark 2.31, the functions $r_{ij}^k : G \to \mathbb{C}$ are linearly independent as functions on $G$, so that for all $s \in G$,
\[
0 = \ell(\tilde{\rho}_s) = \ell(\rho_s^1, \cdots, \rho_s^h) = \sum_{k=1}^h \sum_{i,j} \ell(r_{ij}^k(s) E_{ij}^k) = \sum_{i,j,k} r_{ij}^k(s) \ell(E_{ij}^k) \implies \sum_{i,j,k} \ell(E_{ij}^k) r_{ij}^k = 0
\]
where the last equality is an equality of functions on $G$. It follows that the coefficients $\ell(E_{ij}^k)$ must all be zero, i.e., $\ell = 0$, a contradiction. So $\tilde{\rho}$ is surjective; we use a vector space dimension argument again to conclude bijectivity. (See [1].)

Another proof is as follows: assume $\sum_{s \in G} a_s s$ maps to zero, that is, $\sum_{s \in G} a_s s$ acts as 0 on each of the $W_i$’s. Therefore it must act as zero on all representations by linearity; this is not true since it doesn’t act as zero on $1 \in \mathbb{C}[G]$. Therefore $\tilde{\rho}$ is injective, so we can apply a dimension argument again. (See [2].)

More abstractly, one can reformulate (iii) as follows: if $\rho^i : G \to \text{GL}(W_i)$, $i = 1, \cdots, h$ are the distinct irreducible representations of $G$, then the construction of Theorem 3.8 (iii) gives an isomorphism of $\mathbb{C}$-algebras
\[
\tilde{\rho} : \mathbb{C}[G] \to \prod_{i=1}^h \text{End}_\mathbb{C}(W_i), \quad \tilde{\rho}_s \overset{\text{def}}{=} (\rho_s^1, \cdots, \rho_s^h).
\]

A natural question now arises. Since $\tilde{\rho}$ is an isomorphism, the inverse map is well-defined. What is it? The answer is provided by Corollary 3.11. To proceed, we study the inner product space $\mathbb{C}[G]$ (c.f. Definition 2.24).

**Proposition 3.10.** (Plancherel’s formula) Let $u, v : G \to \mathbb{C}$ be complex functions on $G$. Write $u = \sum_{s \in G} u_s s$. Define
\[
\hat{u} \overset{\text{def}}{=} \sum_{s \in G} u_{s^{-1}} s = \sum_{s \in G} u_s s^{-1}.
\]
Note that $(-) : \mathbb{C}[G] \to \mathbb{C}[G]$ is a $\mathbb{C}$-linear map since it equals pre-composition with the inversion map of $G$. We have the formula
\[
(u, v) = \frac{1}{g} \sum_{i=1}^h n_i \text{tr} (\tilde{\rho}(\hat{u} v)) = \frac{1}{g} \sum_{i=1}^h n_i \text{tr} (\tilde{\rho}(u \tilde{\rho}^{-1})),
\]
The center of Proposition 3.12. called the conjugacy classes of $G$.

Consider $\rho : G \to \text{GL}(W_i)$ of order $n_i$, $i = 1, \cdots, h$. Suppose $(u^i)_{1 \leq i \leq h} \in \prod_{i=1}^{h} \text{End}_{\mathbb{C}}(W_i)$ and $u = \sum_{s \in G} u_s s \in \mathbb{C}[G]$ are such that $\tilde{\rho}(u) = (u^i)_{1 \leq i \leq h}$. Then

$$u_s = \frac{1}{g} \sum_{i=1}^{h} n_i \text{tr} \left( \rho_{s}^i u^i \right).$$

In other words,

$$\tilde{\rho}^{-1}(u^1, \cdots, u^h) = \sum_{s \in G} \left( \frac{1}{g} \sum_{i=1}^{h} n_i \text{tr} \left( \rho_{s}^i u^i \right) \right) s.$$

**Proof.** Since $G \subseteq \mathbb{C}[G]$ is an orthonormal basis, we have $u = \sum_{s \in G} (u, s)s$. It suffices to set $v = s \in G$ in Plancherel’s formula:

$$u_s = (u, s) = \frac{1}{g} \sum_{i=1}^{h} n_i \text{tr} \left( \tilde{\rho}(us^{-1}) \right) = \frac{1}{g} \sum_{i=1}^{h} n_i \text{tr} \left( u^i \rho_{s}^{-1} \right) = \frac{1}{g} \sum_{i=1}^{h} n_i \text{tr} \left( \rho_{s}^{-1} u^i \right).$$

### 3.2 The center of $\mathbb{C}[G]$.

Consider

$$Z(\mathbb{C}[G]) \overset{\text{def}}{=} \{ u \in \mathbb{C}[G] \mid \forall u' \in \mathbb{C}[G], \quad uu' = u'u \},$$

called the **center** of $\mathbb{C}[G]$.

**Proposition 3.12.** The center of $\mathbb{C}[G]$ is a free $\mathbb{C}$-vector space with basis given by the following: let the conjugacy classes of $G$ be written as $C_1, \cdots, C_h$ and the basis $\{ z_1, \cdots, z_h \}$ of $Z(\mathbb{C}[G])$ by

$$z_i \overset{\text{def}}{=} \sum_{s \in C_i} s.$$

In other words, $Z(\mathbb{C}[G]) = \mathbb{C} C_1[G]$ since $z_i : G \to \mathbb{C}$ is the function equal to 1 on $C_i$ and zero elsewhere.

**Proof.** Suppose $\sum_{s \in G} a_s s \in Z(\mathbb{C}[G])$. Then for all $t \in G$, we have

$$\sum_{s \in G} a_s s = t \left( \sum_{s \in G} a_s s \right) t^{-1} = \sum_{s \in G} a_s (ts^{-1}) = \sum_{s \in G} a_{t^{-1}st}s,$$

doing so that $a : G \to \mathbb{C}$ which sends $s \mapsto a_s$ is a class function, hence $\sum_{s \in G} a_s s \in \langle z_1, \cdots, z_h \rangle \mathbb{C}$. Conversely, the elements $z_i$ are in $Z(\mathbb{C}[G])$ by definition of a conjugacy class (they commute with elements of $G$, hence with elements of $\mathbb{C}[G]$ by linearity), so $Z(\mathbb{C}[G]) = \langle z_1, \cdots, z_h \rangle \mathbb{C}$. The linear independence of the $z_i$’s is clear since conjugacy classes are disjoint.
Corollary 3.13. The dimension of $Z(\mathbb{C}[G])$ is the number of conjugacy classes of $G$ and the number of irreducible representations of $G$.

Proposition 3.14. Recall the map $\tilde{\rho} : \mathbb{C}[G] \to \prod_{i=1}^{h} \text{End}_\mathbb{C}(W_i)$ from the proof of Theorem 3.8, which was the product of the homomorphisms $\tilde{\rho}^i : \mathbb{C}[G] \to \text{End}_\mathbb{C}(W_i)$.

The maps $\tilde{\rho}^i$ send $Z(\mathbb{C}[G])$ into the set of homotheties of $W_i$ (namely, the set of endomorphisms of $W_i$ of the form $\lambda \text{id}_{W_i}$ for $\lambda \in \mathbb{C}$), which defines an algebra homomorphism $\omega_i : Z(\mathbb{C}[G]) \to \mathbb{C}$. If $u = \sum_{s \in G} u_s s \in Z(\mathbb{C}[G])$, then by Lemma 2.28, since $u_s$ is a class function on $G$,

$$\omega_i(u) = \frac{1}{n_i} \text{tr}(\tilde{\rho}_i(u)).$$

Furthermore, the family $(\omega_i)_{1 \leq i \leq h}$ defines a $\mathbb{C}$-algebra isomorphism between $Z(\mathbb{C}[G])$ and $\mathbb{C}^h$ (seen as a subset of $\prod_{i=1}^{h} \text{End}_\mathbb{C}(W_i)$) by taking the product of the $\omega_i$’s; the inclusion $\mathbb{C}^h \subseteq \prod_{i=1}^{h} \text{End}_\mathbb{C}(W_i)$ is obtained by considering a vector $(\lambda_1, \ldots, \lambda_h)$ as the element $(\lambda_1 \text{id}_{W_1}, \ldots, \lambda_h \text{id}_{W_h})$.

**Proof.** Since $u \in Z(\mathbb{C}[G])$ maps to a $G$-linear endomorphism $\tilde{\rho}(u)$ of $W_i$, by Corollary 1.33 (noting that $h = h$ in the notation of that corollary), we see that $\tilde{\rho}(u) = \frac{1}{n_i} \text{tr}(\tilde{\rho}(u)) \text{id}_{W_i}$. Note that $\tilde{\rho}|_{Z(\mathbb{C}[G])}$ is injective since $\tilde{\rho}$ is an isomorphism, so the claim follows from $\dim Z(\mathbb{C}[G]) = \dim \mathbb{C}^h$.

### 3.3 Integrality properties of the characters

**Definition 3.15.** Let $R \subseteq S$ be a ring extension, i.e. $R$ and $S$ are both commutative unital rings, $R$ is a subring of $S$ and $1_R = 1_S$. The element $s \in S$ is said to be **integral over** $R$ if there exists $r_1, \ldots, r_n \in R$ such that

$$s^n + r_1 s^{n-1} + \cdots + r_n = 0.$$

We denote the set of all elements $s \in S$ integral over $R$ by $\overline{R}^S$ and call it the **integral closure of $R$ in $S$**. We note that it can be shown that $\overline{R}^S$ is an $R$-algebra, e.g. that $R \subseteq \overline{R}^S$ (which is obvious) and it is closed under addition and multiplication (c.f. Commutative Algebra, Section II.2).

The element $z \in \mathbb{C}$ is said to be an **algebraic integer** if $z \in \overline{\mathbb{Z}}^\mathbb{C}$, the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. This is equivalent to the existence of $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0.$$

**Remark 3.16.** We have $\overline{\mathbb{Z}}^\mathbb{C} \cap \mathbb{Q} = \mathbb{Z}$, e.g. $\mathbb{Z}$ is integrally closed. To see this, if $z \in \overline{\mathbb{Z}}^\mathbb{C} \cap \mathbb{Q}$, write $z = p/q$ with $q \geq 1$, so that $p^n + a_1 p^{n-1} q + \cdots + a_n q^n = 0$; this means $p^n \equiv 0 \pmod{q}$, a contradiction if we take $p$ and $q$ coprime unless $q = 1$, i.e. $p \in \mathbb{Z}$.

**Proposition 3.17.** Let $\chi$ be a character of the complex representation $\rho : G \to \text{GL}(V)$ of the finite group $G$. For all $s \in G$, $\chi(s)$ is an algebraic integer.

**Proof.** $\chi(s)$ is the trace of the linear map $\rho_s$ which is diagonalizable with eigenvalues that are roots of unity, hence algebraic integers; $\chi(s)$ is the sum of these roots, hence $\chi(s)$ is an algebraic integer.

**Proposition 3.18.** Suppose $u = \sum_{s \in G} u_s s \in Z(\mathbb{C}[G])$ is such that $u_s$ is an algebraic integer (notice that $Z(\mathbb{C}[G])$ is a **commutative** unital ring, hence $Z(\mathbb{C}[G]) / \mathbb{Z}$ is a ring extension in the sense of Definition 3.15). Then $u \in \overline{\mathbb{Z}}^{Z(\mathbb{C}[G])}$, i.e. $u$ is integral over $\mathbb{Z}$, so there exists a polynomial with integer coefficients $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$u^n + a_1 u^{n-1} + \cdots + a_n = 0.$$
Proof. Let \( \{z_1, \ldots, z_h\} \) be the \( \mathbb{C} \)-basis given for \( Z(\mathbb{C}[G]) \) in Proposition 3.12. The elements \( z_1, \ldots, z_h \in Z(\mathbb{C}[G]) \) are integral over \( \mathbb{Z} \) since the \( \mathbb{Z} \)-module \( \mathbb{Z}[z_1, \ldots, z_h] \) is finitely generated; to see this, note that \( z_i z_j \in \mathbb{Z}[G] \cap Z(\mathbb{C}[G]) \subseteq \langle z_1, \ldots, z_h \rangle \mathbb{Z} \); c.f. Commutative Algebra, Proposition II.17 and Corollary II.18. Since \( u \) is then a sum of products of integral elements (because \( s \mapsto u_s \) is a class function), we are done, because \( \mathbb{Z}[Z(\mathbb{C}[G])] \) is a subring of \( Z(\mathbb{C}[G]) \).

Corollary 3.19. Let \( G \) be a finite group of order \( g \) and \( \rho \) be an irreducible representation of \( G \) of degree \( n \) with character \( \chi \). If \( u = \sum_{i=1}^h u_i s \in Z(\mathbb{C}[G]) \) (i.e. if \( s \mapsto u_s \) is a class function) and \( u_s \) is an algebraic integer for all \( s \in G \), then

\[
\frac{1}{n} \sum_{s \in G} u_s \chi(s)
\]

is an algebraic integer.

Proof. The number written above is the image under \( \omega_i : Z(\mathbb{C}[G]) \rightarrow \mathbb{C} \) of \( u \), so since \( \omega_i \) is a \( \mathbb{C} \)-algebra isomorphism (where \( \omega_i \) corresponds to \( \chi = \chi_i \), c.f. Proposition 3.14), it is also a \( \mathbb{Z} \)-algebra isomorphism; thus it maps algebraic integers to algebraic integers.

Corollary 3.20. Let \( G \) be a finite group of order \( g \) and \( \chi_1, \ldots, \chi_h \) be the irreducible representations of \( G \) with degrees \( n_1, \ldots, n_h \) respectively. Then \( n_i \) divides \( g \).

Proof. Apply the previous corollary to the element \( u = \sum_{s \in G} \chi_i(s^{-1}) s \in Z(\mathbb{C}[G]) \), which is possible because \( s \mapsto \chi_i(s^{-1}) \) is a class function which maps \( s \) to algebraic integers. Since \( \langle \chi_i, \chi_i \rangle = (\chi_i, \chi_i) = 1 \) by Remark 2.30, we get

\[
g_{n_i} = g(\chi_i, \chi_i)_{n_i} = \frac{1}{n_i} \sum_{s \in G} \chi_i(s) \chi_i(s^{-1}) = \frac{1}{n_i} \sum_{s \in G} \chi_i(s^{-1}) \chi_i(s) = \frac{1}{n_i} \text{tr} (\rho^2(u)) \in \mathbb{Z} \cap \mathbb{Q} = \mathbb{Z},
\]

hence \( n_i \) divides \( g \).

3.4 Frobenius reciprocity

Let \( H \leq G \) be a subgroup of the finite group \( G \) and \( R \) a system of left coset representatives for \( G/H \).

Let \( V \) be a \( \mathbb{C}[G] \)-module, so that \( V \) is also a \( \mathbb{C}[H] \)-module via the inclusion \( \mathbb{C}[H] \subseteq \mathbb{C}[G] \). Let \( W \) be a \( \mathbb{C}[H] \)-submodule of \( V \). Recall that \( V \) is induced by \( W \) if as \( \mathbb{C} \)-vector spaces, we have the following direct sum:

\[
V = \bigoplus_{r \in R} r(W)
\]

Note that the \( r(W) \) are not \( \mathbb{C}[H] \)-submodules of \( V \) in general (c.f. Theorem 3.38 which explains that \( r(W) \) is a \( \mathbb{C}[rH]^{-1} \)-module; in particular, this is a decomposition of \( \mathbb{C}[H] \)-modules if and only if \( H \leq G \) is a normal subgroup). So we insist that this direct sum is merely in the sense of vector spaces. Since we have a \( \mathbb{C}[H] \)-linear inclusion \( i : W \rightarrow V \), we can extend it to a map \( \tilde{i} : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \rightarrow V \).

Remark 3.21. Note that this tensor product is in the sense of non-commutative ring theory, namely that \( \mathbb{C}[G] \) is seen as a \( \mathbb{C}[G] \)-bimodule (both the left and right module action are defined by multiplication in the ring), and thus a \( (\mathbb{C}[G], \mathbb{C}[H]) \)-bimodule. Since \( W \) is a \( \mathbb{C}[H] \)-module, this makes \( \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \) a left \( \mathbb{C}[G] \)-module where for \( s \in G, t \in H \) and \( w \in W \), we have \( st \otimes w = s \otimes \rho^W_t(w) \). This is not the same as taking the tensor product of the two left \( \mathbb{C}[H] \)-modules \( \mathbb{C}[G] \) and \( W \) since in this case, we have \( t \cdot (s \otimes w) = ts \otimes \rho^W_t(w) \) (c.f. Definition 3.1), where as in the left \( \mathbb{C}[G] \)-module \( \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \), the tensor product satisfies \( t \cdot (s \otimes w) = ts \otimes w \).
**Proposition 3.22.** Let $G$ be a finite group, $H$ a subgroup, $V$ a representation of $G$ and $W$ an $H$-subrepresentation of $V$. The following are equivalent:

(i) $V$ is induced by $W$

(ii) There exists an isomorphism of $\mathbb{C}[G]$-modules $V \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$

(iii) The $\mathbb{C}[G]$-linear map $\tilde{\iota}$ is an isomorphism.

**Proof.** The set $R$ is a basis of $\mathbb{C}[G]$ as a $\mathbb{C}[H]$-module, which makes the result clear.

**Definition 3.23.** Let $H$ be a subgroup of the finite group $G$ and $W$ be a representation of $H$. The **induced representation of $G$ by $W$** is defined as $\text{Ind}^{G}_{H}(W) \overset{\text{def}}{=} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$. By the properties of the tensor product, this induces a functor $\text{Ind}^{G}_{H}(-) : \text{Rep}_{\mathbb{C}}(H) \to \text{Rep}_{\mathbb{C}}(G)$ which we call the **induction functor**.

On the other hand, if $V$ is a $\mathbb{C}[G]$-module, one can view it as a $\mathbb{C}[H]$-module. This induces a second functor $\text{Res}^{G}_{H}(-) : \text{Rep}_{\mathbb{C}}(G) \to \text{Rep}_{\mathbb{C}}(H)$ which is forgetful, called the **restriction functor**.

**Remark 3.24.**

- Induction is transitive: if $H \leq K \leq G$ are groups and $W$ is a $\mathbb{C}[H]$-module, then we have natural isomorphisms $\text{Ind}^{G}_{K}(W) \simeq \text{Ind}^{G}_{H}(\text{Ind}^{K}_{H}(W)).$

This is because as $\mathbb{C}[G]$-modules, we have a natural isomorphism given by the properties of the tensor product $\mathbb{C}[G] \otimes_{\mathbb{C}[K]} (\mathbb{C}[K] \otimes_{\mathbb{C}[H]} W) \simeq (\mathbb{C}[G] \otimes_{\mathbb{C}[K]} \mathbb{C}[K]) \otimes_{\mathbb{C}[H]} W \simeq \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$

- Since we constructed an explicit representation of $V$ which is induced by $W$, the induced representation exists and is unique up to isomorphism by the previous proposition. Proposition 3.22 shows more than proving the existence of the induced representation; it shows that it is unique up to a **unique** isomorphism which is natural in $W$, namely $\tilde{\iota}$. This suggests that the induction functor must represent something; in fact, Theorem 3.25 shows that $\text{Ind}^{G}_{H}(W)$ represents the functor $\text{Hom}_{\mathbb{C}[H]}(-, \text{Res}^{G}_{H}(V)) : \text{Rep}_{\mathbb{C}}(H) \to \text{Set}$.

**Theorem 3.25.** Let $H$ be a subgroup of the finite group $G$, $W$ be a representation of $H$ and $V$ be a representation of $G$. We have a natural isomorphism of complex vector spaces $\text{Hom}_{\mathbb{C}[G]}(\text{Ind}^{G}_{H}(W), V) \simeq \text{Hom}_{\mathbb{C}[H]}(W, \text{Res}^{G}_{H}(V))$ given by $\varphi \mapsto (w \mapsto \varphi(1 \otimes w))$. In fact, since $H$ acts on both hom-sets trivially, this is also an isomorphism of $\mathbb{C}[H]$-modules (c.f. Remark 2.17).

**Proof.** By the tensor-hom adjunction for bimodules over non-commutative rings (see Commutative Algebra, Theorem 7.20 for a proof in the commutative case; in the non-commutative case, the proof is the same, except that we must act on the appropriate side (e.g. left or right)), since $\text{Res}^{G}_{H}(V) = V_{\mathbb{C}[H]}$ is the $\mathbb{C}[G]$-module $V$ seen as a $\mathbb{C}[H]$-module, we obtain the natural isomorphisms $\text{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, V) \simeq \text{Hom}_{\mathbb{C}[H]}(\mathbb{C}[H], \text{Hom}_{\mathbb{C}[H]}(W, V_{\mathbb{C}[H]})) \simeq \text{Hom}_{\mathbb{C}[H]}(W, \text{Res}^{G}_{H}(V)).$

Note that the first isomorphism sends a $\mathbb{C}[G]$-linear map $\varphi : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$ to the map $\psi : \mathbb{C}[H] \to \text{Hom}_{\mathbb{C}[H]}(W, V)$ defined by $1 \mapsto (w \mapsto \varphi(1 \otimes w))$, which we extend to all of $\mathbb{C}[H]$ by $\mathbb{C}[H]$-linearity of $\varphi$ (note that $\varphi$ is $\mathbb{C}[G]$-linear). The second isomorphism sends a map $\psi : \mathbb{C}[H] \to \text{Hom}_{\mathbb{C}[H]}(W, V)$ to its value at $1 \in \mathbb{C}[H]$, so the composition of those two isomorphisms sends $\varphi$ to the map $w \mapsto \varphi(1 \otimes w)$. 
Chapter 3

Corollary 3.26. (Universal property of the induced representation) Let $H$ be a subgroup of the finite group $G$, $W$ a representation of $H$ and $V$ a representation of $G$. The induced representation is characterized up to a unique isomorphism by the following property: an $H$-linear map $\varphi : W \to \text{Res}_H^G(V)$ lifts uniquely to a $G$-linear map $\tilde{\varphi} : \text{Ind}_H^G(W) \to V$. Explicitly, $\tilde{\varphi}$ is given by the following formula for $s \in G$ and $w \in W$:

$$\tilde{\varphi}(s \otimes w) \overset{\text{def}}{=} \rho_s^V(\varphi(w)).$$

Proof. Since $\varphi \in \text{Hom}_{C[H]}(W, \text{Res}_H^G(V))$, the map $\tilde{\varphi}$ is just the image of $\varphi$ in $\text{Hom}_{C[G]}(\text{Ind}_H^G(W), V)$ given by Theorem 3.25; to see this, it suffices to notice that $\tilde{\varphi}$ is $G$-linear and agrees with $\varphi$ on $C[H] \otimes_{C[H]} W$ (since $G$ is $C[H]$-linear). The result follows since $\varphi \mapsto \tilde{\varphi}$ is bijective.

Definition 3.27. Let $H$ be a subgroup of the finite group $G$ of order $g$. It is not hard to see that if $V$ is a $C[G]$-module with character $\chi$, then $\text{Res}_H^G(V)$ has character $\text{Res}_H^G(\chi) \overset{\text{def}}{=} \chi|_H$, i.e. the function $\chi : G \to C$ restricted to $H$. The formula in Proposition 2.67 defines $\text{Ind}_H^G(\psi)$ when $\psi$ is the character of a $C[H]$-module $W$. In general, for a function $\psi : H \to C$, we set

$$\text{Ind}_H^G(\psi)(s) \overset{\text{def}}{=} \frac{1}{g} \sum_{t \in G} \psi(t^{-1}st).$$

When $\psi$ is the character of a $C[H]$-module $W$, $\text{Ind}_H^G(\psi)$ is the character of $\text{Ind}_H^G(W)$; also, when $\psi \in C_{cl}[G]$ is a class function, we recover the formula in Proposition 2.67 for $\text{Ind}_H^G(\psi)$ in terms of a system of left coset representatives $R$. For $\varphi_1, \varphi_2 \in C[G]$ two functions on $G$, define

$$\langle \varphi_1, \varphi_2 \rangle_G \overset{\text{def}}{=} \frac{1}{g} \sum_{s \in G} \varphi_1(s)\varphi_2(s^{-1}).$$

(c.f. Remark 2.30).

Theorem 3.28. (Frobenius reciprocity) Let $H$ be a subgroup of the finite group $G$. If $\psi \in C_{cl}[H]$ and $\varphi \in C_{cl}[G]$, then

$$\langle \text{Ind}_H^G(\psi), \varphi \rangle_G = \langle \psi, \text{Res}_H^G(\varphi) \rangle_H.$$

Proof. Since both $\text{Res}_H^G$ and $\text{Ind}_H^G$ are $C$-linear maps, both sides are $C$-bilinear, hence we can restrict to the case where $\psi$ and $\varphi$ are characters by Theorem 2.32. Let $\psi$ be the character of the $C[H]$-module $W$ and $\varphi$ be the character of the $C[G]$-module $V$. The result follows from the row orthogonality relations, Remark 2.30 and Theorem 3.25:

$$\langle \text{Ind}_H^G(\psi), \varphi \rangle_G = \dim \text{Hom}_{C[G]}(\text{Ind}_H^G(W), V) = \dim \text{Hom}_{C[H]}(W, \text{Res}_H^G(V)) = \langle \psi, \text{Res}_H^G(\varphi) \rangle_H.$$

Proposition 3.29. Let $H$ be a subgroup of the finite group $G$. Let $W$ be a representation of $H$ with character $\chi_W$ and $V$ a representation of $G$ with character $\chi_V$.

(i) We have a natural isomorphism of $C[G]$-modules

$$\text{Ind}_H^G(W) \otimes_C V \simeq \text{Ind}_H^G(W \otimes_C \text{Res}_H^G(V))$$

(note that on both sides, this is a tensor product of representations, c.f. Remark 1.16) together with the corresponding identity on characters

$$\text{Ind}_H^G(\chi_W)\chi_V = \text{Ind}_H^G(\chi_W \text{Res}_H^G(\chi_V))$$

which can be extended to class functions by $C$-bilinearity.

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(ii) Assume \(W\) and \(V\) are irreducible. Let \(h^G\) (resp. \(h^H\)) denote the number of conjugacy classes of \(G\) (resp. \(H\)). We decompose the following representations as a sum of irreducible subrepresentations:

\[
\text{Ind}_H^G(W) \simeq \bigoplus_{i=1}^{h^G} V_i^\oplus m_i, \quad \text{Res}_H^G(V) \simeq \bigoplus_{j=1}^{h^H} W_j^\oplus n_j.
\]

Pick the two subscripts \(i_0, j_0\) such that \(V_{i_0} = V\) and \(W_{j_0} = W\). Then

\[
m_{i_0} = \langle \text{Ind}_H^G(\chi_W), \chi_V \rangle_G = \langle \chi_W, \text{Res}_H^G(\chi_V) \rangle_H = n_{j_0}.
\]

**Proof.** (i) An easy proof of this statement is done by computing the characters on both sides, although this proof does not make the isomorphism natural: for \(s \in G\),

\[
\chi_{\text{Ind}_H^G(W) \otimes V}(s) = \chi_{\text{Ind}_H^G(W)}(s) \chi_V(s)
\]

\[
= \left( \frac{1}{h} \sum_{t \in G} \chi_W(t^{-1}st) \right) \chi_V(s)
\]

\[
= \frac{1}{h} \sum_{t \in G} \chi_W(t^{-1}st) \chi_V|_H(t^{-1}st)
\]

\[
= \frac{1}{h} \sum_{t \in G} \chi_W \otimes \text{Res}_H^G(V)(t^{-1}st)
\]

\[
= \chi_{\text{Ind}_H^G(W \otimes \text{Res}_H^G(V))}(s).
\]

For a natural isomorphism, recall Corollary 3.26 so that the canonical map

\[
\varphi : W \otimes \text{Res}_H^G(V) \to \text{Ind}_H^G(W) \otimes V, \quad w \otimes v \mapsto (1 \otimes w) \otimes v
\]

lifts to a \(\mathbb{C}[G]\)-linear map \(\tilde{\varphi} : \text{Ind}_H^G(W \otimes \text{Res}_H^G(V)) \to \text{Ind}_H^G(W) \otimes V\). Explicitly, if \(\rho^V : G \to \text{GL}(V)\) denotes the representation of \(V\), for \(s \in G\), \(w \in W\) and \(v \in V\), we have

\[
\tilde{\varphi}(s \otimes (w \otimes v)) \overset{\text{def}}{=} s \cdot ((1 \otimes w) \otimes v) = (s \otimes w) \otimes \rho^V_v(v).
\]

The map \(\tilde{\varphi}\) is surjective since \(\tilde{\varphi}(s \otimes (w \otimes \rho^V_{s^{-1}}(v))) = (s \otimes w) \otimes v\). Since both vector spaces have the same dimension, we conclude that \(\tilde{\varphi}\) is a \(\mathbb{C}[G]\)-linear isomorphism.

We finish this section with a generalization of the notion of induction representation.

**Definition 3.30.** Let \(\alpha : H \to G\) be a morphism of groups, \(\rho^W : H \to \text{GL}(W)\) a representation of \(H\) and \(\rho^V : G \to \text{GL}(V)\) a representation of \(G\).

(i) The **restriction of \(V\) along \(\alpha\)** is denoted by \(\text{Res}_H^G(V)\). As a vector space, \(\text{Res}_H^G(V) \overset{\text{def}}{=} V\). The morphism of groups \(\text{Res}_H^G(\rho^V) : H \to \text{GL}(V)\) is defined by \(\text{Res}_H^G(\rho^V) \overset{\text{def}}{=} \rho^V \circ \alpha\).

(ii) The **induced representation of \(W\) along \(\alpha\)** is denoted by \(\text{Ind}_\alpha^G(W)\). The group algebra \(\mathbb{C}[G]\) becomes a \(\mathbb{C}[H]\)-algebra via the morphism of rings

\[
\mathbb{C}[\alpha] : \mathbb{C}[H] \to \mathbb{C}[G], \quad \mathbb{C}[\alpha] \left( \sum_{h \in H} a_h h \right) \overset{\text{def}}{=} \sum_{h \in H} a_h \alpha(h).
\]
The induced representation is then defined by extension of scalars of the \( \mathbb{C}[H] \)-module \( W \), namely

\[
\text{Ind}_G^G(W) \overset{\text{def}}{=} \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.
\]

It becomes a \( \mathbb{C}[G] \)-module via the definition \( s \cdot (t \otimes w) \overset{\text{def}}{=} (st) \otimes w \), which for \( h \in H \) satisfies \( \alpha(h) \otimes w = 1 \otimes \rho_h(w) \).

**Theorem 3.31.** Let \( \alpha : H \to G \) be a morphism of groups. Let \( W \) a \( \mathbb{C}[H] \)-module with character \( \chi_W \) and \( V \) a \( \mathbb{C}[G] \)-module with character \( \chi_V \).

(i) We have a natural isomorphism

\[
\text{Hom}_{\mathbb{C}[G]}(\text{Ind}_G^G(W), V) \simeq \text{Hom}_{\mathbb{C}[H]}(W, \text{Res}_H^G(V)).
\]

The map is given by sending \( \varphi : \text{Ind}_G^G(W) \to V \) to the map \( w \mapsto \varphi(1 \otimes w) \) and extending by \( \mathbb{C}[G] \)-linearity.

(ii) (Universal property of the induced representation along \( \alpha \)) The isomorphism given in (i) satisfies the following universal property: if \( \varphi : W \to V \) is a \( \mathbb{C} \)-linear map satisfying \( \varphi(h \cdot w) = \alpha(h) \cdot \varphi(w) \) for all \( h \in H \) (e.g. \( \varphi : W \to \text{Res}_H^G(V) \) is a morphism of \( \mathbb{C}[H] \)-modules), there exists a unique morphism of \( \mathbb{C}[G] \)-modules \( \tilde{\varphi} : \text{Ind}_G^G(W) \to V \) satisfying \( \tilde{\varphi}(h \otimes w) = \alpha(h) \varphi(w) \) for all \( h \in H \) and \( w \in W \).

**Proof.** This is exactly the same proof as in Theorem 3.25, the only difference being that the morphism \( \mathbb{C}[\alpha] : \mathbb{C}[H] \to \mathbb{C}[G] \) is not necessarily injective anymore (but that changes nothing to the proof).

**Corollary 3.32.** Let \( K \overset{\alpha}{\to} H \overset{\beta}{\to} G \) be two morphism of groups. We have the identities of functors

\[
\text{Res}_K^\alpha \circ \text{Res}_H^\beta = \text{Res}_K^{\beta \circ \alpha}, \quad \text{Ind}_K^G \circ \text{Ind}_H^G \simeq \text{Ind}_K^{\beta \circ \alpha}.
\]

**Proof.** The first equality follows from definition. For the second, given representations \( \rho^V : G \to \text{GL}(V) \) and \( \rho^W : K \to \text{GL}(W) \), since we have a natural bijection:

\[
\text{Hom}_{\mathbb{C}[G]}(\text{Ind}_G^G(\text{Ind}_H^H(W)), V) \simeq \text{Hom}_{\mathbb{C}[H]}(\text{Ind}_G^G(W), \text{Res}_H^G(V))
\]

\[
\simeq \text{Hom}_{\mathbb{C}[K]}(W, \text{Res}_K^G(\text{Res}_H^G(V)))
\]

\[
= \text{Hom}_{\mathbb{C}[K]}(W, \text{Res}_K^{\beta \circ \alpha}(V)),
\]

the universal property of the induced representation along \( \beta \circ \alpha \) gives the natural isomorphism.

**Remark 3.33.** There are three particular cases to distinguish about these two notions of restriction and induction along a morphism \( \alpha : H \to G \):

(i) when \( \alpha : H \to G \) is the inclusion map of a subgroup, in which case we recover the classical notion;

(ii) when \( \alpha \overset{\text{def}}{=} \pi_K : G \to G/K \) is the projection modulo a normal subgroup. Recall Section 2.7 concerning lifts and drops of representations. In this case, if \( \rho^V : G/K \to \text{GL}(V) \) and \( \rho^W : G \to \text{GL}(W) \) are representations, then \( \text{Res}_{\pi_K}^G(\rho^V) = L(\rho^V) \) equals the lift of \( \rho^V \) to \( G/K \) along \( \pi_K \). The left-adjoint to \( \text{Res}_{\pi_K}^G \), namely \( \text{Ind}_{\pi_K}^G \), is not equal to the drop in general (c.f. Section 2.7). To understand what it is, let \( W^K \leq W \) be the vector subspace of elements \( w \in W \) invariant under the action of \( K \), i.e. \( \rho_s(w) = w \) for all \( s \in K \) (c.f. Theorem 2.13). Given \( w \in W \), we see that in
\[ \text{Ind}_{\pi}^{G/K}(W) = \mathbb{C}[G/K] \otimes_{\mathbb{C}[G]} W, \]

\[ K \otimes w = \frac{1}{\text{ord}(K)} \sum_{s \in K} sK \otimes w = 1 \otimes \left( \frac{1}{\text{ord}(K)} \sum_{s \in K} \rho_s(w) \right), \]

so that any simple tensor admits an equivalent expression with \( w \in W^K \). Since \( K \) acts trivially on the subspace \( W^K \) and \( K \) is a normal subgroup of \( G \), we deduce that \( W^K \) is a \( G \)-subrepresentation of \( W \), for if \( s \in G, t \in K \) and \( w \in W^K \), letting \( t' \overset{\text{def}}{=} s^{-1}ts \in K \) so that \( ts = st' \), we have

\[ \rho_t(\rho_s(w)) = \rho_s(\rho_t(w)) = \rho_s(w) \implies \rho_s(w) \in W^K. \]

The map \( W^K \to \text{Ind}_{\pi}^{G/K}(W) \) sending \( w \) to \( 1 \otimes w \) thus maps a basis to a basis and is \( G/K \)-linear, hence is an isomorphism of \( \mathbb{C}[G/K] \)-modules.

In particular, if \( W \) is an irreducible representation of \( G \), since \( W^K \) is a \( G \)-subrepresentation, either \( K \) acts trivially on \( W \) (in which case \( \text{Ind}_{\pi}^{G/K}(W) = D(W) \) equals the drop) or \( W^K = \text{Ind}_{\pi}^{G/K}(W) = 0 \). So in this sense, the functor \( \text{Ind}_{\pi}^{G/K} \) is the correct functorial generalization of the drop \( D(\cdot) \) to cover the cases where \( K \) acts non-trivially ; if we decompose \( W \) as a direct sum of irreducible \( K \)-subrepresentations, \( \text{Ind}_{\pi}^{G/K} \) kills those on which \( K \) acts non-trivially and applies \( D(\cdot) \) on the remaining ones.

It is also worth noting that a morphism of groups \( \alpha : H \to G \) can be factored as the surjective morphism \( \alpha : H \to \text{im} \alpha \) followed by the inclusion \( \text{im} \alpha \to G \), so that understanding these two cases allows understanding the general case.

(iii) when \( H \leq G \) is a subgroup, \( s \in G, H^s \overset{\text{def}}{=} sHs^{-1} \) and \( \alpha : H^s \to H \) is the map equal to conjugation by \( s^{-1} \), e.g. \( \alpha(\text{sh} s^{-1}) = s^{-1}(\text{sh} s^{-1})s = h \). Given a representation \( \rho : H \to \text{GL}(W) \) and \( s \in G \), the \( s \)-twist of \( \rho \) is defined by \( \rho^s \overset{\text{def}}{=} \text{Res}_{H^s}^{H}(\rho) : H^s \to \text{GL}(W^s) \) where as vector spaces, \( W^s \overset{\text{def}}{=} W \) and for \( w \in W, h \in H \), we have

\[ \rho^s_h(w) \overset{\text{def}}{=} \rho_{\alpha(h)}(w) = \rho_{s^{-1}hs}(w). \]

By Corollary 3.32, it is clear that \( \text{Res}_{H^s}^{H} \) and \( \text{Res}_{H^s}^{H^{-1}} \) are inverse of each other since \( \text{Res}_{H^s}^{H} \) and \( \text{Res}_{H^s}^{H^{-1}} \) are the identity functors by definition ; therefore, their adjoints \( \text{Ind}_{H^s}^{H} \) and \( \text{Ind}_{H^{-1}s}^{H} \) are also inverse of each other. What is perhaps more interesting is that we have natural isomorphisms \( \text{Res}_{H^s}^{H} \simeq \text{Ind}_{H^{-1}s}^{H} \) (and analogously \( \text{Res}_{H^s}^{H^{-1}} \simeq \text{Ind}_{H^{-1}s}^{H} \)) since if \( \rho : H \to \text{GL}(W) \) is a representation of \( H \) and \( \rho^s : H^s \to \text{GL}(W^s) \) is the corresponding \( s \)-twist, then \( \text{Ind}_{H}^{H}(\rho^s) = \mathbb{C}[H] \otimes_{\mathbb{C}[H_s]} W^s \) is a \( \mathbb{C}[H] \)-module satisfying the following for \( h \in H \) and \( w \in W \):

\[ h \otimes w = \alpha(\alpha^{-1}(h)) \otimes w = 1 \otimes \rho_{\alpha^{-1}(h)}^s(w) = 1 \otimes \rho_{\alpha(\alpha^{-1}(h))}(w) = 1 \otimes \rho_h(w), \]

so that we obtain a natural isomorphism \( \mathbb{C}[H] \otimes_{\mathbb{C}[H_s]} W^s \simeq W, \)

\[ \text{Ind}_{H}^{H} \circ \text{Res}_{H^s}^{H} \simeq \text{id}_{\text{Rep}(H)} \simeq \text{Res}_{H^{-1}s}^{H^{-1}} \circ \text{Res}_{H^s}^{H}, \]

which leads to the following definition.

**Definition 3.34.** Let \( H \) be a subgroup of the finite group \( G \) and \( s \in G \). If \( \rho : H \to \text{GL}(W) \) is a representation of \( H \), its \( s \)-twist is the representation of \( H^s \overset{\text{def}}{=} sHs^{-1} \) given by \( \text{Tw}^s_H(\rho) : H^s \to \text{GL}(W^s) \) where \( \text{Tw}^s_H(\rho) \overset{\text{def}}{=} \text{Res}_{H^s}^{H}(\rho) \simeq \text{Ind}_{H^{-1}s}^{H^{-1}}(\rho) \), where \( \alpha : H^s \to H \) is the map which conjugates by \( s^{-1} \), e.g. \( h \mapsto s^{-1}hs \), and \( W_s \overset{\text{def}}{=} W \) is the underlying vector space which is turned into a \( \mathbb{C}[H^s] \)-module via \( \text{Tw}^s_H(\rho) \).

If \( W \) has character \( \chi \), the character of the twist is obviously given by the formula \( \text{Tw}^s_H(\chi)(t) = \chi(s^{-1}ts) \) for \( t \in H^s \) (note that \( \chi(s^{-1}ts) \neq \chi(t) \) in general since it is possible that \( t \notin H \), hence \( \chi(t) \) is not even defined).
Proposition 3.35. Let $H$ be a subgroup of the finite group $G$ and $s \in G$. If $V$ is a representation of $G$ induced by the $H$-subrepresentation $W$, then $\text{Tw}^G_H(V)$ is induced by the $H^s$-subrepresentation $\text{Tw}^G_H(W)$.

**Proof.** Let $\alpha : G \to G$ denote conjugation by $s^{-1}$ (namely $t \mapsto s^{-1}ts$), so that $\alpha|_H : H^s \to H$ is the map of Definition 3.34. Let $\iota : H \to G$ be inclusion map of $G$ and $\iota^s : H^s \to G$ be the inclusion map of $H^s$. We have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & G \\
\uparrow{\iota} & & \uparrow{\iota} \\
H^s & \xrightarrow{\alpha|_H} & H
\end{array}
$$

which by Corollary 3.32 implies

$$
\text{Ind}_{H^s}^G \circ \text{Tw}^G_H \simeq \text{Ind}_{H^s}^G \circ \text{Ind}_{\alpha^{-1}}^H \simeq \text{Ind}_{\alpha^{-1}}^G \circ \text{Ind}_{H}^G \simeq \text{Tw}_{G}^s \circ \text{Ind}_{H}^G.
$$

Theorem 3.36. Let $\alpha : H \to G$ be a morphism of groups. Let $W$ a $\mathbb{C}[H]$-module with character $\chi$ and $V$ a $\mathbb{C}[G]$-module with character $\psi$. The characters $\text{Res}_H^G(\psi)$ of the representation $\text{Res}_H^G(V)$ and $\text{Ind}_{\alpha}^G(\chi)$ of the representation $\text{Ind}_{\alpha}^G(W)$ satisfy the following formulas :

(i) For $t \in H$, we have $\text{Res}_H^G(\psi)(t) = \psi(\alpha(t))$.

(ii) If $\alpha$ is injective, letting $h \overset{\text{def}}{=} \text{ord}(H)$ and $\alpha^{-1} : \text{im}\alpha \to H$ be the inverse map,

$$
\text{Ind}_{\alpha}^G(\chi)(s) = \frac{1}{h} \sum_{t \in G} \chi(\alpha^{-1}(u)).
$$

(iii) If $\alpha$ is surjective, let $N \overset{\text{def}}{=} \text{ker}\alpha$ and $n \overset{\text{def}}{=} \text{ord}(N)$, so that

$$
\text{Ind}_{\alpha}^G(\chi)(s) = \frac{1}{n} \sum_{z \in H} \chi(z).
$$

(iv) For $\alpha : H \to G$ arbitrary, following the notations of (ii) and (iii), we obtain

$$
\text{Ind}_{\alpha}^G(\chi)(s) = \frac{1}{h} \sum_{t \in G, u \in H} \chi(u).
$$

**Proof.** (i) In the case of restriction, since $\rho^V : G \to \text{GL}(V)$ satisfies $\text{Res}_H^G(\rho^V)(t) = \rho^V_{\alpha(t)}$, we can take the trace and obtain the formula.

(ii) This is just a reformulation of the formula given in Proposition 2.67 to adapt to the case where $\alpha$ is not necessarily the inclusion map of a subgroup since letting $K \overset{\text{def}}{=} \text{im}\alpha$ (which is isomorphic to $H$ via $\alpha : H \to K$), we have

$$
\text{Ind}_{\alpha}^G(\chi)(s) = \text{Ind}_K^K(\text{Ind}_\alpha^K(\chi))(s) = \frac{1}{h} \sum_{t \in G} \text{Ind}_K^K(\chi)(t^{-1}st) = \frac{1}{h} \sum_{t \in G} \chi(\alpha^{-1}(t^{-1}st))
$$

by Remark 3.33.
(iii) Without loss of generality, assume $\alpha : G \to G/K$ is the projection map and $K \leq G$ is a normal subgroup (so that $n \overset{\text{def}}{=} \text{ord}(K)$). In light of Remark 3.33, we consider the $G$-subrepresentation $W^K$ of $W$ isomorphic to $\text{Ind}_G^G(W)$, so that $\chi_{W^K} = \text{Ind}_G^G(\chi)$. By Theorem 2.13, the map $\frac{1}{n} \sum_{t \in K} \rho_t^W : W \to W$ is a projection with image $W^K$. Forming a basis of $W$ by taking the union of a basis of $W^K$ and a basis of $\ker \pi$, we can evaluate the following trace for $s \in G$ as

$$
\chi_{W^K}(sK) = \text{tr} \left( \rho_s^W|_{W^K} \right) = \frac{1}{n} \sum_{t \in K} \text{tr} \left( \rho_s^W \circ \rho_t^W \right) = \frac{1}{n} \sum_{t \in K} \chi(st) = \frac{1}{n} \sum_{z \in G \atop \alpha(z) = s} \chi(z).
$$

(iv) Let $N \overset{\text{def}}{=} \ker \alpha$ and $n \overset{\text{def}}{=} \text{ord}(N)$, so we can write $\alpha$ as the composition $H \overset{\pi_N}{\longrightarrow} H/N \overset{\tilde{\alpha}}{\longrightarrow} G$ where $\tilde{\alpha}$ is injective. Let $h \overset{\text{def}}{=} \text{ord}(H)$, so that in both cases, we have the formula for the induced character by (ii) and (iii), hence we evaluate : for $s \in G$,

$$
\text{Ind}_G^H(\chi)(s) = \text{Ind}_G^H \left( \text{Ind}_{H/N}^G(\chi) \right)(s) = \frac{1}{\text{ord}(H/N)} \sum_{t \in G, u \in H/N \atop t^{-1}st = \tilde{\alpha}(u)} \text{Ind}_{H/N}^G(\chi)(uN) = \frac{n}{h} \sum_{t \in G, u \in H \atop t^{-1}st = \alpha(u)} \left( \frac{1}{n} \sum_{z \in H \atop \pi_N(z) = uN} \chi(z) \right) = \frac{1}{h} \sum_{t \in G, u \in H \atop t^{-1}st = \alpha(u)} \chi(u).
$$

**Theorem 3.37.** (Frobenius reciprocity along $\alpha$) Let $\alpha : H \to G$ be a morphism of groups, $\varphi \in \mathbb{C}_{\text{cl}}[H]$ and $\psi \in \mathbb{C}_{\text{cl}}[G]$. Then

$$
\langle \text{Ind}_G^H(\varphi), \psi \rangle_G = \langle \varphi, \text{Res}_H^G(\psi) \rangle_H.
$$

**Proof.** By $\mathbb{C}$-bilinearity, we can restrict to the case of characters ; the result then follows from Theorem 3.31 by taking dimensions.

### 3.5 Group actions and induced representations

**Theorem 3.38.** Suppose $V$ is a $\mathbb{C}[G]$-module such that

$$
V \overset{\text{def}}{=} \bigoplus_{i \in I} W_i
$$

where the $W_i$’s are vector subspaces which are permuted by $G$, so that for any $s \in G$ and $i \in I$, there exists a unique $i' \in I$ satisfying $\rho_s(W_i) = W_{i'}$. Let $G \acts I$ be defined as the unique action characterized by the equation $\rho_s(W_i) = W_{s\cdot i}$ for all $i \in I$.

(i) For $i \in I$, let $H_i \overset{\text{def}}{=} \text{Stab}_G(W_i)$. If $G$ acts transitively on $I$, then $V$ is induced by the subrepresentation...
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$W_i$ of $H_i$. Moreover, if $s \in G$ and $j = s \cdot i$, then $H_j = sH_is^{-1}$.

(ii) If we do not assume that the action $G \cap I$ is transitive, recall that $G \setminus I$ denotes the set of orbits of $I$. Given $i \in J \subset G \setminus I$ (so that $J = \text{Orb}_G(i)$), set

$$V_J \overset{def}{=} \bigoplus_{j \in J} W_j$$

as vector spaces. By construction, since $G \cap J$ is transitive, $V_J$ is a $G$-subrepresentation of $V$ induced by the $H_i$-subrepresentation $W_i$, e.g. $V_J = \text{Ind}_{H_i}^G(W_i)$. As representations of $G$, we obtain

$$V = \bigoplus_{J \in G \setminus I} V_J.$$

**Proof.** (i) Since $H_i = \text{Stab}_G(W_i) = \text{Stab}_G(i)$, for $s \in G$, any $t \in H_i$ gives $\rho_{st}(W_i) = \rho_s(W_{t,i}) = \rho_s(W_i)$, so that if $R$ is a system of left coset representatives for $G/H$, we have

$$V = \bigoplus_{j \in I} W_j = \bigoplus_{r \in R} r(W_i).$$

From Definition 2.38, we deduce

$$H_j = \text{Stab}_G(j) = \text{Stab}_G(s \cdot i) = s\text{Stab}_G(i)s^{-1} = sH_is^{-1}.$$

It is clear that

(ii) This follows from (i) ; we simply recorded the statement.

**Definition 3.39.** Let $\Lambda$ be a partition of $\{1, \cdots, n\}$, i.e. $\Lambda$ is a set of non-empty disjoint subsets of $\{1, \cdots, n\}$ whose union equals $\{1, \cdots, n\}$. Given an $n \times n$ matrix $A = (a_{ij})$, the matrix block corresponding to $(\sigma, \sigma') \in \Lambda \times \Lambda$ is the matrix formed of the coefficients $a_{ij}$ where $(i, j) \in \sigma \times \sigma'$ and is denoted by $A^{\sigma, \sigma'}$. If $f$ is the linear endomorphism of $\mathbb{C}^n$ associated to $A$, then the linear map $f^{\sigma, \sigma'}$ (which we also call a matrix block) corresponding to the matrix $A^{\sigma, \sigma'}$ is the map

$$f^{\sigma, \sigma'} : \langle\{e_j\}_{j \in \sigma'}\rangle_{\mathbb{C}} \rightarrow \langle\{e_i\}_{i \in \sigma}\rangle_{\mathbb{C}}$$

defined by the coefficients of $A^{\sigma, \sigma'}$.

Suppose $\rho : G \rightarrow \text{GL}(\mathbb{C}^n)$ is a complex matrix representation and let $\Lambda$ be a partition of the set $\{1, \cdots, n\}$. An element of $\Lambda \times \Lambda$ is called a block. We say that $\rho$ is in block form with respect to $\Lambda$ if for each $\sigma \in \Lambda$ and each $s \in G$, there are unique $\sigma', \sigma'' \in \Lambda$ such that the matrix blocks $\rho_s^{\sigma, \sigma'}, \rho_s^{\sigma'', \sigma}$ are non-zero (in other words, one non-zero block per row and column). Note that it suffices to make the check on rows (resp. columns) since $\rho_s$ is an isomorphism, thus has maximal rank (and so a row or column of zeros is impossible).

Saying that $\rho$ is in block form means that if $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{C}^n$, we have a direct sum decomposition

$$\mathbb{C}^n = \bigoplus_{\sigma \in \Lambda} W_\sigma, \quad W_\sigma \overset{def}{=} \langle\{e_i\}_{i \in \sigma}\rangle_{\mathbb{C}}$$

and for every $s \in G$ and $\sigma' \in \Lambda$, $\rho_s(W_{\sigma'}) \subseteq W_\sigma$ where $\sigma \in \Lambda$ is unique with the property that $\rho_{s}^{\sigma', \sigma} \neq 0$. This suggests that we can define an action $G \cap \Lambda$ by setting $s \cdot \sigma' = \sigma$ if $\rho_s(W_{\sigma'}) \subseteq W_\sigma$. We call this the block form action corresponding to $\rho : G \rightarrow \text{GL}(\mathbb{C}^n)$. 60
Corollary 3.40. We reformulate Theorem 3.38 in block form using Definition 3.39. Let $\Lambda$ be a partition of $\{1, \cdots, n\}$ and $V = \mathbb{C}^n$ be the representation corresponding to the morphism of groups $\rho : G \to \text{GL}(V)$ which is in block form with respect to $\Lambda$.

(i) Suppose the corresponding action $G \circ \Lambda$ is transitive. Let $H_{\sigma} = \text{Stab}_G(\sigma)$ and consider the $H_{\sigma}$-subrepresentation $\theta$ of $\mathbb{C}^n$ given by the matrices $\theta_s \overset{\text{def}}{=} \rho_s \sigma$. Then $\rho$ is induced by $\theta$ and if $\sigma, \sigma'$ satisfy $\sigma = s \cdot \sigma'$, then $H_{\sigma} = s H_{\sigma'} s^{-1}$. Furthermore, for any $\sigma, \sigma' \in \Lambda$, we have $|\sigma| = |\sigma'|$.

(ii) Given $\sigma' \in \Omega \subset G \setminus \Lambda$ (so that $\Omega = \text{Orb}_G(\sigma')$), set

$$V_{\Omega} \overset{\text{def}}{=} \bigoplus_{\sigma \in \Omega} W_\sigma \subseteq V,$$

as vector spaces. By construction, since the restricted action $G \circ \Omega$ is transitive, $V_{\Omega}$ is a representation of $G$ induced by the $H_{\sigma'}$-subrepresentation $W_{\sigma'}$, e.g. $V_{\Omega} = \text{Ind}_{H_{\sigma'}}^G(W_{\sigma'})$. As representations of $G$, we obtain

$$V = \bigoplus_{\Omega \in G \setminus \Lambda} V_{\Omega}.$$

In other words, to decompose a representation in block form into subrepresentations of $G$, one can look at the orbits of the block form action and decompose as a direct sum of the underlying vector spaces spanned by the blocks, which will be subrepresentations; furthermore, these subrepresentations are induced from the subgroups fixing a block in an orbit.

Proof. (i) The only thing we have not proven yet is that $|\sigma| = |\sigma'|$, which is obvious since $W_\sigma \simeq W_{\sigma'}$ via $\rho_s$ where $s \cdot \sigma' = \sigma$, so that

$$|\sigma| = \dim W_\sigma = \dim W_{\sigma'} = |\sigma'|.$$

(ii) This entirely follows from interpreting Theorem 3.38.

Remark 3.41. Let $G \circ X$ be a group action (both sets are finite). Consider the associated permutation representation $\rho : G \to \text{GL}(V)$ where $V = \{\langle e_x \rangle_{x \in X}\} \mathbb{C}$ (c.f. Example 1.5). By definition, $\rho$ satisfies the hypotheses of Theorem 3.38 with $V = \bigoplus_{x \in X} W_x$ where $W_x \overset{\text{def}}{=} \langle e_x \rangle \mathbb{C}$ since $\rho_s(W_x) = W_{s \cdot x}$. It follows that

(i) if $G \circ X$ is transitive and $H_x \overset{\text{def}}{=} \text{Stab}_G(x)$ for $x \in X$, then $V$ is induced by the $H_x$-subrepresentation $W_x$ and for $s \in G$, $H_{s \cdot x} = H_x s^{-1}$

(ii) given $A \subseteq G \setminus X$ and setting

$$V_A \overset{\text{def}}{=} \{\langle e_x \rangle_{x \in A}\} \mathbb{C},$$

we have a direct sum decomposition of $V \simeq \bigoplus_{A \subseteq G \setminus X} V_A$ into $G$-subrepresentations and for each $x \in A \subseteq G \setminus X$, the $G$-subrepresentation $V_A$ is induced by the trivial representation of $H_x$ (since $W_x$ is 1-dimensional and $H_x$ acts trivially on $W_x$ by definition).

Example 3.42. Consider the classical representation $\rho : S_n \to \text{GL}(V)$ associated to the transitive group action $S_n \circ \{1, \cdots, n\}$. We show that it is induced by the trivial representation of $S_{n-1}$ (c.f. Example 1.5), where $S_{n-1}$ is seen as the stabilizer of the element $n \in \{1, \cdots, n\}$. Writing $V = \langle e_1, \cdots, e_n \rangle \mathbb{C}$ and $W_i \overset{\text{def}}{=} \langle e_i \rangle \mathbb{C}$ for $1 \leq i \leq n$, we have $V = \bigoplus_{i=1}^n W_i$ and $\rho$ permutes the $W_i$ transitively (because $S_n$ acts transitively), so that $S_{n-1} \simeq H_n = \text{Stab}_{S_n}(n) \leq S_n$ is the subgroup inducing $V$ via its trivial representation.

As for its character $\chi^C$ (C stands for classical), via the formula for the induced character, we see that if

$$R = \{\text{id}_{\{1, \cdots, n\}}, (1n), (2n), \cdots, ((n-1)n)\} = \{(in) \in S_n \mid i \in \{1, \cdots, n\}\}$$

we have

$$\chi^C(\text{id}_{\{1, \cdots, n\}}) = \chi_{\{1, \cdots, n\}}.$$

Therefore, $\chi^C$.

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is our chosen system of left coset representatives for \( S_n/S_{n-1} \), for \( \sigma \in S_n \), since \( \chi^C \) is induced by the trivial character of \( S_{n-1} \), we have

\[
\chi^C(\sigma) = \sum_{\pi \in \mathbb{R}} 1 \quad \pi^{-1}\sigma \pi \in S_{n-1}
\]

\[
= \# \{ i \in \{1, \ldots, n\} | (\sigma \circ (in))(n) = (in)(n) \}
\]

\[
= \# \{ i \in \{1, \ldots, n\} | \sigma(i) = i \},
\]

as predicted by the formula given in Proposition 2.41.

**Example 3.43.** Let \( H \leq G \) be a subgroup of the finite group \( G \). Since the group action \( G \circ G/H \) is transitive, the corresponding permutation representation \( \mathbb{C}[G/H] \) can be decomposed as

\[
\mathbb{C}[G/H] = \bigoplus_{\sigma \in G/H} W_\sigma, \quad W_\sigma \overset{def}{=} \langle e_\sigma \rangle \mathbb{C}
\]

where \( s \in G \) permutes the \( W_\sigma \) transitively since \( \rho_s(W_\sigma) = W_{\sigma \sigma} \) by definition. It is clear that \( \text{Stab}_G(W_H) = H \), showing that \( \mathbb{C}[G/H] \) is induced by the trivial representation of \( H \) (because \( H \) acts trivially on \( W_H \)) by Theorem 3.38. We had seen this directly in Example 2.68.

**Example 3.44.** Recall Example 2.71, where we computed the irreducible characters of the dihedral group \( D_n \). Supposedly we do not know what the irreducible representation \( \rho : D_n \to \text{GL}_2(\mathbb{C}) \) could look like and we want to construct it (note that \( n > 2 \), since for \( n = 2 \), \( D_2 \) is abelian). The matrix \( \rho_r \) is diagonalizable by Theorem 2.11, so we can already write

\[
\rho_r = \begin{bmatrix} \omega^{i_1} & 0 \\ 0 & \omega^{i_2} \end{bmatrix}
\]

where \( \omega \overset{def}{=} e^{2\pi i/n} \) is a primitive \( n \)th root of unity and \( i_1, i_2 \in \{1, n-1\} \) (since these are the only two generators of the cyclic group \( \mathbb{Z}/n\mathbb{Z} \)). We know that \( \rho_s \) cannot be diagonalizable over the eigenbasis of \( \rho_r \), otherwise \( \rho \) would be reducible. Since \( \rho_{srs^{-1}} = \rho_r^{-1} \), letting \( \{v_1, v_2\} \) be the eigenbasis of \( \rho_r \) satisfying \( \rho_r(v_j) = \omega^{i_j} \), we can write

\[
\alpha_1 \omega^{-i_1} v_1 + \alpha_2 \omega^{-i_2} v_2 = \rho_r^{-1}(\alpha_1 v_1 + \alpha_2 v_2) = \rho_r(\rho_s(v_1)) = \rho_s(\rho_r(v_1)) = \rho_s(\omega^{i_1} v_1) = \omega^{i_1}(\alpha_1 v_1 + \alpha_2 v_2)
\]

which implies \( \alpha_1(\omega^{i_1} - \omega^{-i_1}) = 0 \) and \( \alpha_2(\omega^{i_1} - \omega^{-i_2}) = 0 \). The first equation implies \( \alpha_1 = 0 \) and \( \alpha_2 \neq 0 \) (because \( \rho_s(v_1) = \alpha_1 v_1 + \alpha_2 v_2 \neq 0 \)), so that the second equation implies \( \omega^{i_1} = \omega^{-i_2} \), i.e. \( i_2 = n-i_1 \). It follows that \( \rho_s(v_1) = \pm v_2 \), and by applying \( \rho_s \) on that equation, we deduce that \( \rho_s(v_2) = \mp v_1 \). Letting \( i_1 = 1 \) and choosing the + sign for \( \rho_s \) gives the representation

\[
\rho_r = \begin{bmatrix} \omega^{i_1} & 0 \\ 0 & \omega^{i_2} \end{bmatrix}, \quad \rho_s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

which we can see as being induced by \( \langle r \rangle \simeq \mathbb{Z}/n\mathbb{Z} \) by Corollary 3.40 (where \( \Lambda = \{\{1\}, \{2\}\} \)).

### 3.6 Restriction to subgroups; Mackey’s irreducibility criterion

In this section, three groups are constantly involved, so for simplicity, the letters \( k \) and \( h \) will denote group elements instead of group orders or number of conjugacy classes of \( G \).

Let \( H, K \leq G \) be two subgroups. Then we can partition \( G \) using the \( (H, K)\)-double cosets of \( G \):

\[
KsH \overset{def}{=} \{ksh | k \in K, h \in H \}.
\]
This is because we have a left action $K \times H \circ G$ given by $(k, h) \circ s \overset{\text{def}}{=} ksh^{-1}$, which we call the **double-coset action** of the pair $(K, H)$. The subset $KsH$ is simply the orbit of $s \in G$ under the action of $K \times H$; we call such an orbit a $(K, H)$-**double coset**. Let $S \subseteq G$ denote a fixed set of representatives of $(K, H)$-double cosets for $K \backslash G / H$. This notation makes sense since this orbit space is also the orbit space of the action $K \circ G / H$ given by multiplication on the left and the orbit space of the right action $K \backslash G \circ H$ given by multiplication on the right (in the above, we artificially turned this action into a left action).

**Theorem 3.45.** Let $H, K \leq G$ be two subgroups of the finite group $G$, $S$ a system of $(K, H)$-double coset representatives for $K \backslash G / H$ and $\rho : H \to \text{GL}(W)$ a representation. Set $H_s \overset{\text{def}}{=} Hs \cap K = sHs^{-1} \cap K$. We have an isomorphism of $\mathbb{C}[K]$-modules:

$$\text{Res}^G_K(\text{Ind}^G_H(W)) \simeq \bigoplus_{s \in S} \text{Ind}^K_{H_s}(\text{Res}^H_{H_s}(\text{Tw}^s_H(W))).$$

**Proof.** Let $V \overset{\text{def}}{=} \text{Ind}^G_H(W)$ with corresponding representation $\rho^V : G \to \text{GL}(V)$. Note that for $w \in W$ and $h \in H$, $\rho^V_h(w) = \rho_h(w)$ since $W$ is an $H$-subrepresentation of $V$ by definition. Consider the canonical decomposition of the $K$-representation $\text{Res}^G_K(V)$ (c.f. Definition 2.35):

$$\text{Res}^G_K(V) = \bigoplus_{s \in S} V_s, \quad V_s \overset{\text{def}}{=} \sum_{k \in K} \rho^V_k(s).$$

To see why this sum is direct, remember that $V$ is induced by $W$, thus $V$ is the direct sum of the $\rho_s(W)$ with $r$ ranging over a system $R$ of left coset representatives for $G / H$. By definition of $S$, each $r \in R$ satisfies $r = ks$ for a unique $s \in S$ and some $k \in K$. The vector space $V_s$ is then the sum over those $r$ which correspond to $(k, s)$ for some $k \in K$, e.g. the sum over all of $k$ of the $\rho^V_k(s)$.

It remains to show that $V_s \simeq \text{Ind}^K_{H_s}(\text{Res}^H_{H_s}(\text{Tw}^s_H(W)))$. Clearly, the subgroup of those $k \in K$ satisfying $\rho^V_k(\rho^V_s(W)) = \rho_s(W)$ equals $H^s_k = sHs^{-1} \cap K$, hence $V_s$ is induced by $\text{Res}^H_{H_s}(\rho_s(W))$. A $\mathbb{C}[H^s]$-isomorphism $\text{Tw}^s_H(W) \simeq \rho_s(W)$ is given by $\Phi_s \overset{\text{def}}{=} \lambda_s : W \to \rho^s_V(W)$ since for $t \in H^s$, we have

$$\Phi_s(t \cdot w) = \rho^V_s(\text{Tw}^s_H(\rho_s(w)) = \rho_s(\rho_s^{-1}t\rho_s(w)) = \rho_s^V(\rho_s(w)) = t \cdot \Phi_s(w).$$

In Theorem 3.45, we can set $K \overset{\text{def}}{=} H$; it follows that $H_s \overset{\text{def}}{=} H_s^H \overset{\text{def}}{=} sHs^{-1} \cap H = H^s \cap H$ for all $s \in G$. Write $\rho : H \to \text{GL}(W)$. For $s \in S$ where $S$ is a set of $(H, H)$-double coset representatives for $H \backslash G / H$, we get two representations of $H_s$, namely $\text{Res}^H_{H_s}(\text{Tw}^s_H(\rho))$ and $\text{Res}^H_{H_s}(\rho)$. In general, these two are not isomorphic.

**Definition 3.46.** Let $V_1, V_2$ be two representations of $G$ with characters $\chi_1, \chi_2$, respectively. We say that they are **linearly disjoint** if $\langle \chi_1, \chi_2 \rangle_G = 0$, i.e. if $V_1$ and $V_2$ have no irreducible components in common (equivalently, if the inner product in $\mathbb{C}[G]$ satisfies $\langle \chi_1 \mid \chi_2 \rangle = 0$).

**Corollary 3.47.** (Mackey’s criterion) Let $H \leq G$ and $W$ a representation of $H$. Let $S$ be a set of $(H, H)$-double cosets representatives of $G$, and without loss of generality assume $1 \in S$. Then $V \overset{\text{def}}{=} \text{Ind}^G_H(W)$ is irreducible if and only if $W$ is irreducible and for all $s \in G \backslash H$, the $H_s$-representations $\text{Res}^H_{H_s}(\text{Tw}^s_H(\rho))$ and $\text{Res}^H_{H_s}(\rho)$ are linearly disjoint.

**Proof.** Let $\chi_W$ denote the character of $W$. We know that $V$ is irreducible if and only if its character $\chi_V = \text{Ind}^G_H(\chi_W)$ satisfies $\langle \chi_V, \chi_V \rangle_G = 1$. Using Frobenius reciprocity (twice, once on the pair $(G, H)$
and then on \((H, H_s)\), recalling that \((-\,-\) is commutative,
\[
\langle \chi_V, \chi_V \rangle_H = \langle \chi_W, \text{Res}_H^G(\chi_V) \rangle_H \\
= \sum_{s \in S} \langle \chi_W, \text{Ind}_{H_s}^H(\text{Res}_H^G(\text{Tw}_H^s(\chi_W))) \rangle_H \\
= \sum_{s \in S} \langle \text{Res}_H^{H_s}(\chi_W), \text{Res}_H^{H_s}(\text{Tw}_H^s(\chi_W)) \rangle_{H_s} \\
= \langle \chi_W, \chi_W \rangle_H + \sum_{s \in S \setminus \{1\}} \langle \text{Res}_H^{H_s}(\chi_W), \text{Res}_H^{H_s}(\text{Tw}_H^s(\chi_W)) \rangle_{H_s}.
\]

Since the numbers involved at both ends are non-negative integers and \(\langle \chi_W, \chi_W \rangle_H > 0\), the result follows.

**Corollary 3.48.** Suppose \(H \trianglelefteq G\) is a normal subgroup and \(W\) is a representation of \(H\). Then \(V \overset{\text{def}}{=} \text{Ind}_H^G(W)\) is irreducible if and only if \(W\) is irreducible and \(\rho\) is not isomorphic to \(\text{Tw}_H^s(\rho)\) for all \(s \in G/H\).

**Remark 3.49.** Let \(H \leq G\) be a subgroup of index \(|G:H| = 2\); in particular \(H \trianglelefteq G\). Pick \(s \in G \setminus H\) so that \(G = H \cup sH\). Let \(\rho : H \to \text{GL}(W)\) be a representation of \(H\). There are two possibilities: either \(\rho \simeq \text{Tw}_H^s(\rho)\) or \(\rho \not\simeq \text{Tw}_H^s(\rho)\).

If \(\rho \simeq \text{Tw}_H^s(\rho)\), then \(\text{Ind}_H^G(\text{Ind}_H^G(W)) = W \oplus \text{Ind}_H^G(W_s) \simeq W \oplus W\); in particular, \(\text{Ind}_H^G(W)\) is a reducible representation of \(G\) by Corollary 3.48. As an example, take \(W\) to be an irreducible representation of \(H\) and let \(G = H \times \mathbb{Z}/2\mathbb{Z}\); write \(\mathbb{Z}/2\mathbb{Z} = \{1, s\}, h \overset{\text{def}}{=} (h, 1)\) and \(s \overset{\text{def}}{=} (1, s)\) so that \(\text{Res}_H^G(\text{Ind}_H^G(W)) \simeq W \oplus sW\) is the decomposition given by Theorem 3.45; furthermore, we see that as \(\mathbb{C}[H]\)-modules, \(sW \simeq W\). We also see that \(\text{Ind}_H^G(W)\) is a reducible representation of \(G\) since
\[
\text{Ind}_H^G(W) \simeq \mathbb{C}[H \times \mathbb{Z}/2\mathbb{Z}] \otimes_{\mathbb{C}[H]} W \simeq (1 + s)W \oplus (1 - s)W
\]
and these two representations are the box products (c.f. Definition 1.22) of the representation \(W\) with the irreducible representations of \(G/H \simeq \mathbb{Z}/2\mathbb{Z}\). We note that the trivial representation and the alternating representation are not isomorphic, hence so is their tensor product with \(W\) (c.f. Theorem 2.57). Worth noting is that when \(V\) is reducible, the decomposition of Theorem 3.45 is in general not the decomposition of \(V\) into irreducible subrepresentations.

If \(\rho \not\simeq \rho^s\), then \(\text{Ind}_H^G(\rho)\) is an irreducible representation of \(G\) by Corollary 3.48. For example, if we induce the representations of the cyclic group \(C_n\) defined by \(\psi_h(r^k) = \omega^k_h\) to \(D_n\) (c.f. Example 2.71), since \(|D_n:C_n| = 2\), we are in the right context. The characters satisfy \(\text{Tw}_{C_n}^r(\psi_h) = \psi_h\) if and only if \(\omega_h = \omega_h^{-1}\), i.e. if and only if \(h\) is divisible by \(n/2\), which coincides again with our previous results (note that representations of degree 1 are their own characters).

**Example 3.50.** Let \(G = S_5\) and \(H = A_5\), so that \(|G:H| = 2\). Recall the computation of the character table of \(A_5\) made in Example 2.72:

<table>
<thead>
<tr>
<th>(c_s)</th>
<th>1</th>
<th>20</th>
<th>15</th>
<th>12</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_5)</td>
<td>1</td>
<td>(123)</td>
<td>(12)(34)</td>
<td>(12345)</td>
<td>(12354)</td>
</tr>
<tr>
<td>(\chi_1)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\chi_2)</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(\chi_3)</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>(\frac{1 + \sqrt{5}}{2})</td>
<td>(\frac{1 - \sqrt{5}}{2})</td>
</tr>
<tr>
<td>(\chi_4)</td>
<td>3</td>
<td>0</td>
<td>-1</td>
<td>(\frac{1 - \sqrt{5}}{2})</td>
<td>(\frac{1 + \sqrt{5}}{2})</td>
</tr>
<tr>
<td>(\chi_5)</td>
<td>5</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Let $s \overset{def}{=} (45)$ be the chosen representative for the non-trivial $(A_5,A_5)$-double coset of $S_5$. For $i = 1,2,5$, we see that $\chi_i(s^{-1}s) = \chi_i(s)$ for all $\sigma \in S_5$, hence it follows that $\chi_i = T_w^{A_5}(\chi_i)$, so $\text{Ind}_{A_5}^{S_5}(\chi_i)$ is reducible by Corollary 3.48. For $i = 3,4$, we see that $\chi_i(s^{-1}(12345)s) = \chi_i((123)(45)) \neq \chi_i((12345))$, hence the characters

$$\psi_5(\sigma) \overset{def}{=} \text{Ind}_{A_5}^{S_5}(\chi_3)(\sigma) = \begin{cases} \chi_3(\sigma) + \chi_3(s^{-1}s) & \text{if } \sigma \in A_5 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi'_5(\sigma) \overset{def}{=} \text{Ind}_{A_5}^{S_5}(\chi_4)(\sigma) = \begin{cases} \chi_4(\sigma) + \chi_4(s^{-1}s) & \text{if } \sigma \in A_5 \\ 0 & \text{otherwise} \end{cases}$$

are irreducible by Corollary 3.48. We note that $\psi_5 = \psi'_5$. As a sidenote, one easily checks (using the completed character table) that

$$\text{Ind}_{A_5}^{S_5}(\chi_1) = 2\psi_1 \quad \text{Ind}_{A_5}^{S_5}(\chi_2) = 2\psi_3 \quad \text{Ind}_{A_5}^{S_5}(\chi_5) = \psi_6 + \psi_7$$

$$\text{Res}_{A_5}^{S_5}(\psi_1) = \chi_1 \quad \text{Res}_{A_5}^{S_5}(\psi_3) = \chi_2 \quad \text{Res}_{A_5}^{S_5}(\psi_6) = \text{Res}_{A_5}^{S_5}(\psi_7) = \chi_5$$

We can compute the character table of $S_5$ by using the trivial/alternating/standard representation for $\psi_1/\psi_2/\psi_3$, multiplying $\psi_4 = \psi_2\psi_3$ and using the induced representations $\text{Ind}_{A_5}^{S_5}(\chi_3) = \psi_5 = \text{Ind}_{A_5}^{S_5}(\chi_4)$ (note that the characters $\chi_3$ and $\chi_4$ both induce $\psi_5$ since the only conjugacy classes of $A_5$ on which $\chi_3$ and $\chi_4$ differ form a single conjugacy class on $S_5$). Since there are 7 conjugacy classes, we are missing 2 irreducible characters. Since

$$120 = |S_5| = \sum_{i=1}^7 n_i^6 = 1 + 1 + 16 + 16 + 36 + n_6^2 + n_7^2 \implies n_6^2 + n_7^2 = 50$$

there are two possibilities for $(n_6, n_7)$ up to permutation, namely (5, 5) and (7, 1). Using the group theory of the symmetric group, since characters of degree 1 are morphisms to an abelian group, they have to contain $[S_5,S_5] = A_5$ in their kernel, hence correspond to characters of $\mathbb{Z}/2\mathbb{Z}$. This means $S_5$ has only two irreducible characters of degree 1, e.g. $(n_6, n_7) = (5,5)$.

We now make the assumption that at least one of the values $\chi_6((12))$, $\chi_6((123)(45))$ or $\chi_6((12345))$ is non-zero (which will turn out to be true), which implies the relationship $\psi_6 \psi_2 = \psi_6 \psi_2$; the hypothesis is necessary to ensure that $\psi_6 \psi_2 \neq \psi_6$, so that it is indeed equal to our remaining character of degree 5. We have the following unknowns in the character table:

<table>
<thead>
<tr>
<th>$c_s$</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>15</th>
<th>20</th>
<th>24</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_5$</td>
<td>1</td>
<td>(12)</td>
<td>(123)</td>
<td>(12)(34)</td>
<td>(123)(45)</td>
<td>(12345)</td>
<td>(1234)</td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_5$</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_6$</td>
<td>5</td>
<td>$x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$x_4$</td>
<td>$x_5$</td>
<td>$x_6$</td>
</tr>
<tr>
<td>$\psi_7$</td>
<td>5</td>
<td>$-x_1$</td>
<td>$x_2$</td>
<td>$x_3$</td>
<td>$-x_4$</td>
<td>$x_5$</td>
<td>$-x_6$</td>
</tr>
</tbody>
</table>

The orthogonality of the pairs of columns $(1,(123))$, $(1,(12)(34))$ and $(1,(12345))$ shows that $x_2 = -1, x_3 = 1$ and $x_5 = 0$; that of the columns $((12),(123)(45))$, $((12),(1234))$ and $((123)(45),(1234))$ gives us the equations $x_1x_4 = 1$, $x_1x_6 = -1$ and $x_4x_6 = -1$, from which we can deduce $x_1 = x_4 = -x_6$ and $x_2^2 = 1$. Giving $x_1$ the values $\pm 1$ gives us the two remaining characters (which we can confirm that they are
irreducible characters by computing $\langle \chi_6, \chi_6 \rangle = 1 = \langle \chi_7, \chi_7 \rangle$, so here is the completed character table of $S_5$:

<table>
<thead>
<tr>
<th></th>
<th>$c_s$</th>
<th>1</th>
<th>10</th>
<th>20</th>
<th>15</th>
<th>20</th>
<th>24</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_5$</td>
<td>(12)</td>
<td>(123)</td>
<td>(1234)</td>
<td>(12345)</td>
<td>(12345)</td>
<td>(1234)</td>
<td></td>
</tr>
<tr>
<td>$\psi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_2$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
</tr>
<tr>
<td>$\psi_3$</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_4$</td>
<td>4</td>
<td>$-2$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_5$</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$-2$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_6$</td>
<td>5</td>
<td>1</td>
<td>$-1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$\psi_7$</td>
<td>5</td>
<td>$-1$</td>
<td>$-1$</td>
<td>1</td>
<td>$-1$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Fun fact: one is welcome to check that $\psi_2^2 = \psi_5$, e.g. the second exterior power of the standard representation $\psi_3$ is the irreducible representation $\psi_5$. 
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<th>Description</th>
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