The discrete Voronoi game

Dániel Gerbner, Viola Mészáros, Dömötör Pálvölgyi, Alexey Pokrovskiy, Günter Rote

Methods for Discrete Structures,
Freie Universität Berlin, Berlin.
alja123@gmail.com

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Competitive facility location

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Example:
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The discrete Voronoi game

The discrete $t$-round Voronoi game is played on the vertices of a graph $G$ with the following rules:

1. Players alternate choosing vertices of $G$ for a fixed number of $t$ rounds.
2. At the end of the game, each player receives:
   - 1 point for every vertex closer to his chosen vertices than his opponent's.
   - $1/2$ point for every vertex equidistant to each player's chosen vertices.
3. The winner is the player with the most points.
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Examples:
The Voronoi ratio

Definition

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- $VR(S_n, t) = 1 - \frac{t}{n}$. 

There are graphs with $VR(G, t) \leq \epsilon$ [Gerbner, Mészáros, Pálvölgyi, P., Rote].
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Bounds on the Voronoi ratio

Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

For every graph $G$ we have

$$\frac{1}{2} VR(G, 1) \leq VR(G, t) \leq \frac{1}{2} (VR(G, 1) + 1).$$
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The upper bound is equivalent to

$$\frac{1}{2} (1 - VR(G, 1)) \leq (1 - VR(G, t)).$$

Thus the theorem can be summarised as “under optimal play in $t$ rounds, either player can claim at least half of what he can in one round.”
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- In general neither bound can be significantly improved.
Bounds on the Voronoi ratio

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For every graph $G$ we have

\[ \frac{1}{2} VR(G, 1) \leq VR(G, t) \leq \frac{1}{2} (VR(G, 1) + 1). \]

The upper bound is equivalent to

\[ \frac{1}{2} (1 − VR(G, 1)) \leq (1 − VR(G, t)). \]

Thus the theorem can be summarised as “under optimal play in $t$ rounds, either player can claim at least half of what he can in one round.”

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For some classes of graphs the above gives good bounds on the $t$-round Voronoi ratio.
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- Both bounds are proved by strategy stealing
Proof sketch

Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

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Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

For every graph $G$ we have

$$\frac{1}{2} \text{VR}(G, 1) \leq \text{VR}(G, t).$$

- Suppose that theorem is false and Player 2 has a strategy to claim more than $1 - \frac{1}{2} \text{VR}(G, 1)$ of the vertices in $t$ rounds.
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- Suppose that theorem is false and Player 2 has a strategy to claim more than $1 - \frac{1}{2} VR(G, 1)$ of the vertices in $t$ rounds.
- Then Player 1’s strategy is:
  - Make the optimal first move in the one round game (which claims $VR(G, 1)$ of the vertices.)
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Then Player 1’s strategy is:

- Make the optimal first move in the one round game (which claims $VR(G, 1)$ of the vertices.)
- Follow Player 2’s optimal strategy (for $t - 1$ moves).
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For every graph $G$ we have

$$\frac{1}{2} \ VR(G, 1) \leq \ VR(G, t).$$

- Suppose that theorem is false and **Player 2** has a strategy to claim more than $1 - \frac{1}{2} VR(G, 1)$ of the vertices in $t$ rounds.

- Then **Player 1**’s strategy is:
  1. Make the optimal first move in the one round game (which claims $VR(G, 1)$ of the vertices.)
  2. Follow **Player 2**’s optimal strategy (for $t - 1$ moves).

- It is possible to show that playing the last move could not gain more that $VR(G, 1)$ of the vertices.
Proof sketch

Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

For every graph \( G \) we have

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\frac{1}{2}(1 - VR(G, 1)) \leq (1 - VR(G, t))
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Theorem (Gerbner, Mészáros, Pálvölgyi, P., Rote)

For every graph $G$ we have

$$\frac{1}{2} (1 - VR(G, 1)) \leq (1 - VR(G, t))$$

Player 2’s strategy:

Player 1 plays the first move, $v$. Player 2 identifies the best response $u$, but does not play it. Instead, he considers the set $S$ of vertices which would be won by playing $u$. Clearly $|S| \geq 1 - VR(G, 1)$. For the rest of the game Player 2 just tries to win the subgame on $S$. 
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Conjecture (P.)

There is an $\alpha > 0$ such that for every planar $G$ we have

$$VR(G, t) \geq \alpha.$$
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Problem

Can bounds on $VR(G, t)$ in terms of $VR(G, 1)$ be improved if $|G| \gg t$. 

Alexey Pokrovskiy (FU Berlin)
Open problems

**Conjecture (P.)**

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Can bounds on $VR(G, t)$ in terms of $VR(G, 1)$ be improved if $|G| \gg t$.

**Problem**

If Player 1 starts by making $k$ simultaneous moves, and then Player 2 makes just one move, then can Player 2 still win 99% of the graph?
Open problems

Conjecture (P.)

There is an $\alpha > 0$ such that for every planar $G$ we have $V_R(G, t) \geq \alpha$.

Problem

Can bounds on $V_R(G, t)$ in terms of $V_R(G, 1)$ be improved if $|G| \gg t$?

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If Player 1 starts by making $k$ simultaneous moves, and then Player 2 makes just one move, then can Player 2 still win 99% of the graph?