Freie Universität Berlin
Institut für Mathematik
Stand: 3. Mai 2024

SoSe24
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Zahlentheorie II – Homework 3

Submission: individually or in pairs,
on Whiteboard as `Names_ZT2_H3.pdf` **by 12:00 on Thursday**, the 9th. of May 2024.

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let $\varphi : A \longrightarrow B$ be a ring homomorphism. Define the **integral closure** of A in B as:

$$\overline{A} = \{b \in B \;\; : \;\; b \text{ is integral over } A\}.$$

1. Show that $\overline{A}$ is a ring.
2. Show that φ restricts to an integral ring homomorphism $A \longrightarrow \overline{A}$.

Problem 2. **2 points**

Let A be a unique factorization domain. Show that $A = \overline{A}$, where $\overline{A}$ is the integral closure of A in its field of fractions $Q(A)$.

Problem 3. **2 points**

Show that every algebraically closed field is infinite.

Problem 4. **2 points**

Let K be a field and $K(X)$ be the field of fractions of the polynomial ring $K[X]$. Prove that $K(X)$ is not algebraically closed.

Total: 8 points

Extra Problem 5.

Let $p > 2$ be prime and ζ be a primitive p-root of unity. Show that the ideal $(1 - \zeta) \subseteq \mathbb{Z}[\zeta]$ is maximal. Which field is $\mathbb{Z}[\zeta]/(1 - \zeta)$?

Extra Problem 6.

Let $A \subset B$ be a ring extension such that $B \setminus A$ is closed under multiplication. Show that A is integrally closed in B.

Extra Problem 7.

Let $\mathbb{K}$ be a field. Does the category of (finitely generated) $\mathbb{K}$-algebras have the epimorphism surjectivity property**?

** i.e. is every epimorphism surjective. A homomorphism $f : A \longrightarrow B$ is an *epimorphism* if it is right cancellative: if for $g, h : C \longrightarrow C$ we have $g \circ f = h \circ f$, then $g = h$.