Generic and special constructions of pure O-sequences

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(Equivalently, it is the f-vector of a pure multicomplex)

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Conjecture′ (Stanley '77) *If* Δ *is a matroid, then* h^Δ *is a pure O-sequence.*

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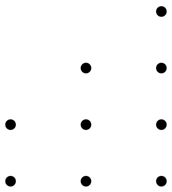

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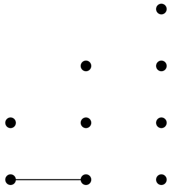

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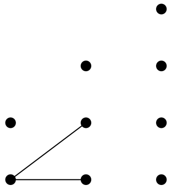

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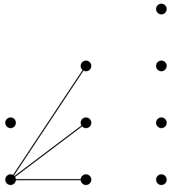

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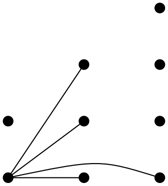

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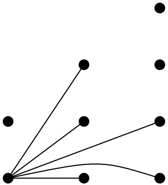

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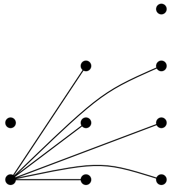

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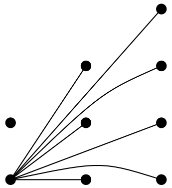

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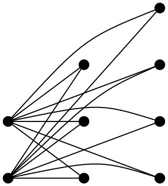

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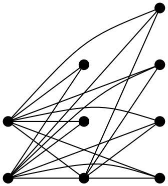

Special strategy: Partition the set of all matroids

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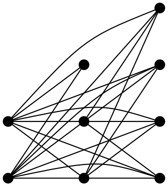

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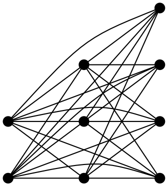

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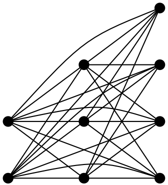

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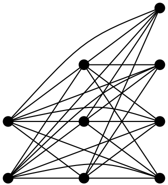

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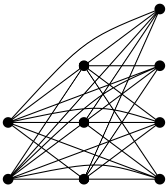

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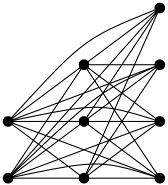

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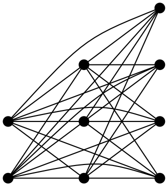

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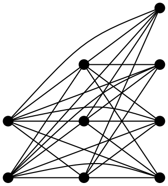

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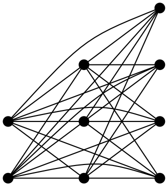

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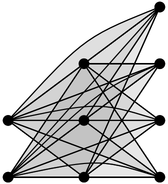

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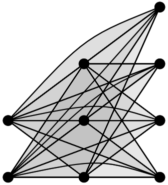

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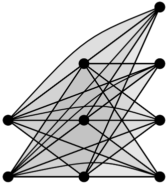

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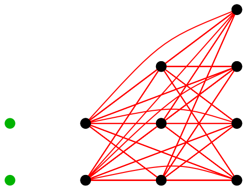

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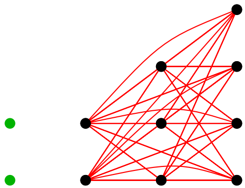

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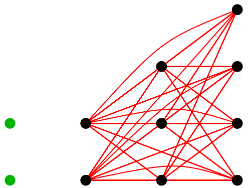

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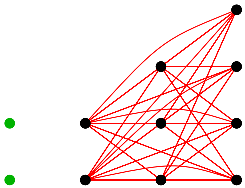

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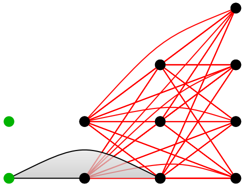

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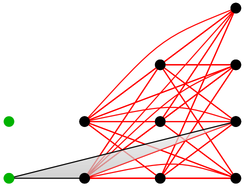

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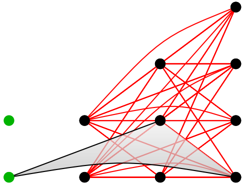

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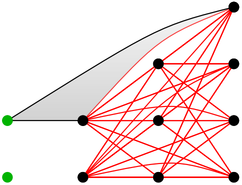

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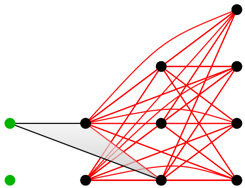

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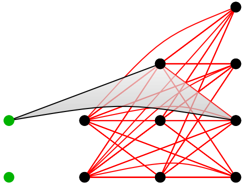
$\forall\ t \in 0..d-2$ define:

$$\Delta_t(d, p, \mathbf{a}) = \Delta_0(t, t, (a_1, \ldots, a_t)) * \Delta_0(d-t, p-t, (a_{t+1}, \ldots, a_p)).$$

Special strategy: Partition the set of all matroids

Lemma *The 1-skeleton Δ^1 of a matroid Δ is a complete p-partite graph.*

$$\Delta^1 = \langle F \;:\; \dim F = 1 \rangle.$$



$\Delta_1(3, 4, (2, 2, 3, 4))$

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For any $d \geq 2$ and any p-partition of n : $\mathbf{a} = (a_1 \leq \cdots \leq a_p)$, we define

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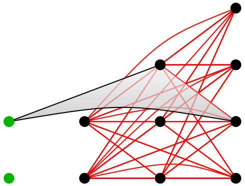

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	$\Delta_0(3,5,(2,2,2,2,3))$	:	1	3	6	10	15	16	24	18	6
Example	Δ	:	h_0	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
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"And now for something completely different"

Generic strategy

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Theorem (Stanley '77) *The Stanley-Reisner ring of a matroid is level.*

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A graded, CM[1] algebra S/I is level if: $0 \longrightarrow S(-a)^{\beta_p} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow S/I \longrightarrow 0$

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Stanley-Reisner ring	S/I_Δ	●	●	●

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Artinian reduction	$S/(I_\Delta + (\ell_i))$	● (green)	● (red)	● (green)
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Artinian reduction	$S/(I_\Delta + (\ell_i))$	● (green)	● (red)	● (green)
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Artinian reduction	$S/(I_\Delta + (\ell_i))$	green	red	green
find a weakgin(I)	$S/(\mathrm{weakgin}(I_\Delta) + (x_i - x_j))$			
Artinian reduction of gin(I)	$S/(\mathrm{gin}(I_\Delta) + (x_i))$	green	green	orange

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Artinian reduction	$S/(I_\Delta + (\ell_i))$	● (green)	● (red)	● (green)
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Thank you for your attention!