Combinatorial Structures, Lefschetz Property and Parametrizations of Algebras

Alexandru Constantinescu

Dipartimento di Matematica, Università di Genova

April 2, 2009

Tesi per il conseguimento del
Dottorato di Ricerca in Matematica e Applicazioni, ciclo XX

Supervisore: Prof. Aldo Conca

THE BIG PICTURE

Three Problems

1. Problems regarding the Veronese algebras and modules of algebras with straightening laws.
2. Characterize the Hilbert function of algebras with m-times the Weak Lefschetz Property (WLP).
3. Parametrize ideals $I \subset k[x, y]$ with $\mathrm{in}(I) = I_0$, when I_0 is the lex-segment ideal.
   Do the same for homogeneous $I \subset k[x, y, z]$ with some extra assumptions.

THE BIG PICTURE
Three Problems

1. Problems regarding the Veronese algebras and modules of algebras with straightening laws.
2. Characterize the Hilbert function of algebras with m-times the Weak Lefschetz Property (WLP).
3. Parametrize ideals $I \subset k[x, y]$ with $\mathrm{in}(I) = I_0$, when I_0 is the lex-segment ideal.
   Do the same for homogeneous $I \subset k[x, y, z]$ with some extra assumptions.

THE BIG PICTURE

Three Problems

1. Problems regarding the Veronese algebras and modules of algebras with straightening laws.
2. Characterize the Hilbert function of algebras with m-times the Weak Lefschetz Property (WLP).
3. Parametrize ideals $I \subset k[x, y]$ with $\mathrm{in}(I) = I_0$, when I_0 is the lex-segment ideal.
   Do the same for homogeneous $I \subset k[x, y, z]$ with some extra assumptions.

THE BIG PICTURE

Three Problems

1. Problems regarding the Veronese algebras and modules of algebras with straightening laws.
2. Characterize the Hilbert function of algebras with m-times the Weak Lefschetz Property (WLP).
3. Parametrize ideals $I \subset k[x, y]$ with $\mathrm{in}(I) = I_0$, when I_0 is the lex-segment ideal.

   Do the same for homogeneous $I \subset k[x, y, z]$ with some extra assumptions.

1. Problems regarding the Veronese algebras and modules of algebras with straightening laws.
2. Characterize the Hilbert function of algebras with m-times the Weak Lefschetz Property (WLP).
3. Parametrize ideals $I \subset k[x, y]$ with $\mathrm{in}(I) = I_0$, when I_0 is the lex-segment ideal.
   Do the same for homogeneous $I \subset k[x, y, z]$ with some extra assumptions.

P A P E R S

1. A. Constantinescu
   Veronese Algebras and Modules of Rings with Straightening Laws,
   Work in progress.
2. A. Constantinescu
   Hilbert Function and Betti Numbers of Algebras with Lefschetz Property of Order m,
   Communications in Algebra, **36**, 12 (2008) , 4704–4720.
3. A. Constantinescu
   Parametrizations of Ideals of $k[x, y]$ *and* $k[x, y, z]$,
   Work in progress.

PAPERS

1. A. Constantinescu
   Veronese Algebras and Modules of Rings with Straightening Laws,
   Work in progress.
2. A. Constantinescu
   Hilbert Function and Betti Numbers of Algebras with Lefschetz Property of Order m,
   Communications in Algebra, **36**, 12 (2008) , 4704–4720.
3. A. Constantinescu
   Parametrizations of Ideals of $k[x, y]$ and $k[x, y, z]$,
   Work in progress.

PAPERS

1. A. Constantinescu
   Veronese Algebras and Modules of Rings with Straightening Laws,
   Work in progress.
2. A. Constantinescu
   Hilbert Function and Betti Numbers of Algebras with Lefschetz Property of Order m,
   Communications in Algebra, **36**, 12 (2008) , 4704–4720.
3. A. Constantinescu
   Parametrizations of Ideals of $k[x, y]$ and $k[x, y, z]$,
   Work in progress.

PAPERS

1. A. Constantinescu
   Veronese Algebras and Modules of Rings with Straightening Laws,
   Work in progress.
2. A. Constantinescu
   Hilbert Function and Betti Numbers of Algebras with Lefschetz Property of Order m,
   Communications in Algebra, **36**, 12 (2008) , 4704–4720.
3. A. Constantinescu
   Parametrizations of Ideals of $k[x, y]$ and $k[x, y, z]$,
   Work in progress.

Is the Veronese algebra of an ASL again an ASL?

Is the Veronese algebra of an ASL again an ASL?

Let k be a field and A a k-algebra. Let $P \subset A$ be a partially ordered set (poset).

We call a monomial a product of elements of P:

$$M = \alpha_1\alpha_2 \dots \alpha_m, \quad \alpha_i \in P.$$

We say that M is a standard monomial if

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m.$$

We call such a weakly increasing sequence of elements of P a m-multichain.

Let k be a field and A a k-algebra. Let $P \subset A$ be a partially ordered set (poset).

We call a monomial a product of elements of P:

$$M = \alpha_1 \alpha_2 \ldots \alpha_m, \quad \alpha_i \in P.$$

We say that M is a standard monomial if

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m.$$

We call such a weakly increasing sequence of elements of P a m-multichain.

Let k be a field and A a k-algebra. Let $P \subset A$ be a partially ordered set (poset).

We call a monomial a product of elements of P:

$$M = \alpha_1 \alpha_2 \ldots \alpha_m, \quad \alpha_i \in P.$$

We say that M is a standard monomial if

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m.$$

We call such a weakly increasing sequence of elements of P a m-multichain.

Let k be a field and A a k-algebra. Let $P \subset A$ be a partially ordered set (poset).

We call a monomial a product of elements of P:

$$M = \alpha_1 \alpha_2 \dots \alpha_m, \quad \alpha_i \in P.$$

We say that M is a standard monomial if

$$\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_m.$$

We call such a weakly increasing sequence of elements of P a m-multichain.

Let k be a field and A a k-algebra. Let $P \subset A$ be a partially ordered set (poset).

We call a monomial a product of elements of P:

$$M = \alpha_1 \alpha_2 \ldots \alpha_m, \quad \alpha_i \in P.$$

We say that M is a standard monomial if

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_m.$$

We call such a weakly increasing sequence of elements of P a m-multichain.

Definition

A k-algebra A is a (graded) algebra with straightening laws on P over k if:

(ASL 0) $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra such that
$A_0 = k$,
P consists of homogeneous elements of positive degree and
P generates A as a k-algebra.

(ASL 1) The standard monomials are a basis of A as a k-vector space.

(ASL 2) (Straightening Laws)
If α and β are incomparable (written $\alpha \not\sim \beta$) and

$$\alpha\beta = \sum r_i \gamma_{i1} \gamma_{i2} \cdots \gamma_{it_i}, \qquad (1)$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \ldots \leq \gamma_{it_i}$
then $\gamma_{i1} < \alpha$ and $\gamma_{i1} < \beta$ for every i.

Definition

A k-algebra A is a (graded) algebra with straightening laws on P over k if:

(ASL 0) $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra such that $A_0 = k$, P consists of homogeneous elements of positive degree and P generates A as a k-algebra.

(ASL 1) The standard monomials are a basis of A as a k-vector space.

(ASL 2) (Straightening Laws) If α and β are incomparable (written $\alpha \not\sim \beta$) and

$$\alpha\beta = \sum r_i \gamma_{i1} \gamma_{i2} \ldots \gamma_{it_i}, \tag{1}$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \ldots \leq \gamma_{it_i}$ then $\gamma_{i1} < \alpha$ and $\gamma_{i1} < \beta$ for every i.

Definition

A k-algebra A is a (graded) algebra with straightening laws on P over k if:

(ASL 0) $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra such that
$A_0 = k$,
P consists of homogeneous elements of positive degree and
P generates A as a k-algebra.

(ASL 1) The standard monomials are a basis of A as a k-vector space.

(ASL 2) (Straightening Laws)
If α and β are incomparable (written $\alpha \not\sim \beta$) and

$$\alpha\beta = \sum r_i \gamma_{i1} \gamma_{i2} \ldots \gamma_{it_i}, \quad (1)$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \ldots \leq \gamma_{it_i}$
then $\gamma_{i1} < \alpha$ and $\gamma_{i1} < \beta$ for every i.

Definition

A k-algebra A is a (graded) algebra with straightening laws on P over k if:

(ASL 0) $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra such that
$A_0 = k$,
P consists of homogeneous elements of positive degree and
P generates A as a k-algebra.

(ASL 1) The standard monomials are a basis of A as a k-vector space.

(ASL 2) (Straightening Laws)
If α and β are incomparable (written $\alpha \not\sim \beta$) and

$$\alpha\beta = \sum r_i \gamma_{i1}\gamma_{i2}\ldots\gamma_{it_i}, \tag{1}$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \ldots \leq \gamma_{it_i}$
then $\gamma_{i1} < \alpha$ and $\gamma_{i1} < \beta$ for every i.

When $P \subset A_1$ we say that A is a homogeneous ASL over P.

An ASL on P for which $\alpha\beta = 0, \forall \alpha \not\sim \beta$ is called the discrete ASL on P.

A subset I of a poset P is called a poset ideal if:

$$\alpha \in I, \ \beta \in P \text{ and } \alpha \geq \beta, \text{ then } \beta \in I.$$

When $P \subset A_1$ we say that A is a homogeneous ASL over P.

An ASL on P for which $\alpha\beta = 0, \forall \alpha \not\sim \beta$ is called the discrete ASL on P.

A subset I of a poset P is called a poset ideal if:

$$\alpha \in I,\ \beta \in P \text{ and } \alpha \geq \beta,\ \text{then } \beta \in I.$$

When $P \subset A_1$ we say that A is a homogeneous ASL over P.

An ASL on P for which $\alpha\beta = 0, \forall \alpha \not\sim \beta$ is called the discrete ASL on P.

A subset I of a poset P is called a poset ideal if:

$$\alpha \in I, \ \beta \in P \text{ and } \alpha \geq \beta, \text{ then } \beta \in I.$$

When $P \subset A_1$ we say that A is a homogeneous ASL over P.

An ASL on P for which $\alpha\beta = 0, \forall \alpha \not\sim \beta$ is called the discrete ASL on P.

A subset I of a poset P is called a poset ideal if:

$$\alpha \in I,\ \beta \in P \text{ and } \alpha \geq \beta,\ \text{then } \beta \in I.$$

We have the following generalization to modules, due to W. Bruns:

Definition

Let A be an ASL on P over k. An A-module M is called module with straightening laws on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1\alpha_2\ldots\alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_i, \ \ i \geq 0,$$

form a basis of M as a k-vector space. These elements are called standard elements.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y < x} Ay. \tag{2}$$

We have the following generalization to modules, due to W. Bruns:

Definition

Let A be an ASL on P over k. An A-module M is called module with straightening laws on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1\alpha_2\ldots\alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_i, \; i \geq 0,$$

form a basis of M as a k-vector space. These elements are called standard elements.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y<x} Ay. \tag{2}$$

We have the following generalization to modules, due to W. Bruns:

Definition

Let A be an ASL on P over k. An A-module M is called module with straightening laws on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1\alpha_2 \dots \alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_i, \quad i \geq 0,$$

form a basis of M as a k-vector space. These elements are called standard elements.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y<x} Ay. \qquad (2)$$

We have the following generalization to modules, due to W. Bruns:

Definition

Let A be an ASL on P over k. An A-module M is called module with straightening laws on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1\alpha_2 \ldots \alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_i, \ \ i \geq 0,$$

form a basis of M as a k-vector space. These elements are called standard elements.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y<x} Ay. \qquad (2)$$

We have the following generalization to modules, due to W. Bruns:

Definition

Let A be an ASL on P over k. An A-module M is called module with straightening laws on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1 \alpha_2 \ldots \alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_i, \;\; i \geq 0,$$

form a basis of M as a k-vector space. These elements are called standard elements.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y < x} A y. \tag{2}$$

Given a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$, and $d \geq 2$ a natural number, the d-Veronese algebra of A is:

$$A^{(d)} = \bigoplus_{i \geq 0} A_{di}.$$

For every $d \geq 2$ and for every $0 \leq j \leq d-1$ the jth Veronese module is

$$M_j^{(d)} = \bigoplus_{i \geq 0} A_{di+j}.$$

The module $M_j^{(d)}$ is obviously an $A^{(d)}$ module.

Given a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$, and $d \geq 2$ a natural number, the d-Veronese algebra of A is:

$$A^{(d)} = \bigoplus_{i \geq 0} A_{di}.$$

For every $d \geq 2$ and for every $0 \leq j \leq d-1$ the jth Veronese module is

$$M_j^{(d)} = \bigoplus_{i \geq 0} A_{di+j}.$$

The module $M_j^{(d)}$ is obviously an $A^{(d)}$ module.

Given a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$, and $d \geq 2$ a natural number, the d-Veronese algebra of A is:

$$A^{(d)} = \bigoplus_{i \geq 0} A_{di}.$$

For every $d \geq 2$ and for every $0 \leq j \leq d-1$ the jth Veronese module is

$$M_j^{(d)} = \bigoplus_{i \geq 0} A_{di+j}.$$

The module $M_j^{(d)}$ is obviously an $A^{(d)}$ module.

Given a graded k-algebra $A = \bigoplus_{i \geq 0} A_i$, and $d \geq 2$ a natural number, the d-Veronese algebra of A is:

$$A^{(d)} = \bigoplus_{i \geq 0} A_{di}.$$

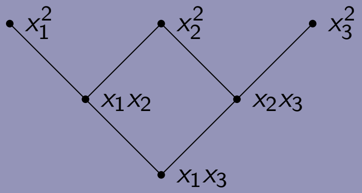
For every $d \geq 2$ and for every $0 \leq j \leq d-1$ the jth Veronese module is

$$M_j^{(d)} = \bigoplus_{i \geq 0} A_{di+j}.$$

The module $M_j^{(d)}$ is obviously an $A^{(d)}$ module.

STRAIGHTENING LAWS

Example

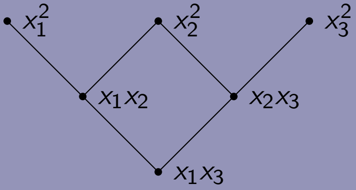


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

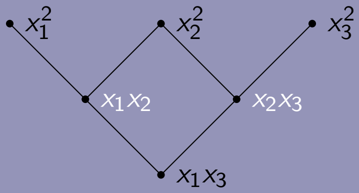
Example



$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

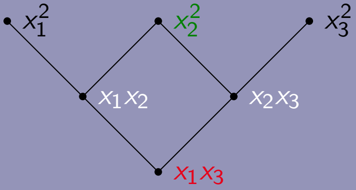
where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS
Example



$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

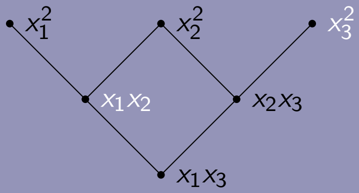


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

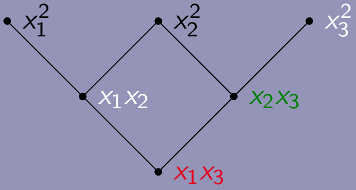


$$\begin{aligned}(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\ (x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\ (x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\ (x_i^2)(x_j^2) &= (x_ix_j)^2,\end{aligned}$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

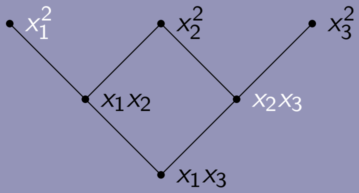


$$\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2), \\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3), \\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2), \\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

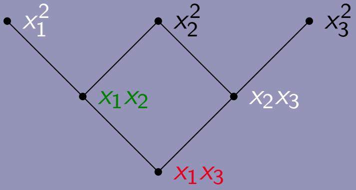


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

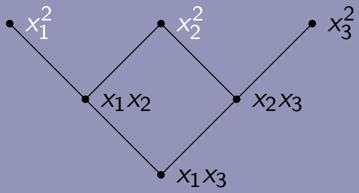


$$\begin{aligned} (x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2), \\ (x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3), \\ (x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2), \\ (x_i^2)(x_j^2) &= (x_ix_j)^2, \end{aligned}$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

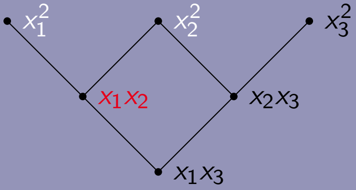


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

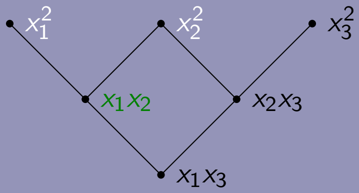


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2), \\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3), \\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2), \\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

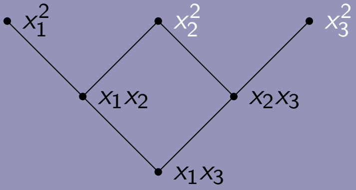


$$\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

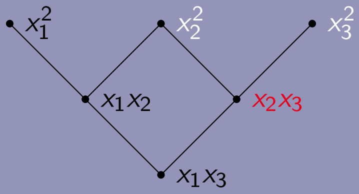


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

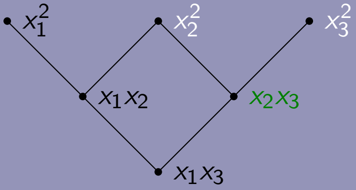


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

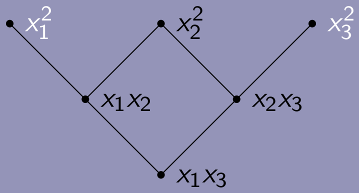


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

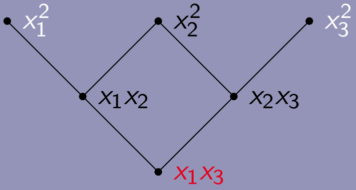


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example

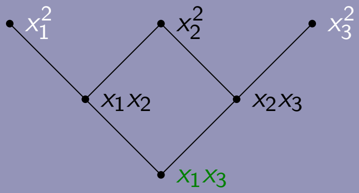


$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2), \\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3), \\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2), \\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

STRAIGHTENING LAWS

Example



$$
\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2), \\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3), \\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2), \\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}
$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

Question: Is the Veronese algebra of an ASL still an ASL?
Answer: We do not know (in general).
Problem: Given a poset P construct a new poset $P^{(d)}$.

Question: Is the Veronese algebra of an ASL still an ASL?

Answer: We do not know (in general).

Problem: Given a poset P construct a new poset $P^{(d)}$.

Question: Is the Veronese algebra of an ASL still an ASL?
Answer: We do not know (in general).
Problem: Given a poset P construct a new poset $P^{(d)}$.

Question: Is the Veronese algebra of an ASL still an ASL?
Answer: We do not know (in general).
Problem: Given a poset P construct a new poset $P^{(d)}$.

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\ j_1, \ldots, j_n \in \overline{1,d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n-d+1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\ j_1, \ldots, j_n \in \overline{1, d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \le \ldots \le x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\, j_1, \ldots, j_n \in \overline{1,d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \le n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \le \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \le t_i, \forall i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\ j_1, \ldots, j_n \in \overline{1, d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\, j_1, \ldots, j_n \in \overline{1,d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall\, i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\ j_1, \ldots, j_n \in \overline{1,d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall\, i.$$

$R = k[x_1, \dots, x_n]$ has an ASL structure: $x_1 \leq \dots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\ j_1, \dots, j_n \in \overline{1,d}, \quad \ell_{1,j_1}, \dots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall\, i.$$

$R = k[x_1, \ldots, x_n]$ has an ASL structure: $x_1 \leq \ldots \leq x_n$.

A. Conca proved that $R^{(d)}$ is still an ASL.

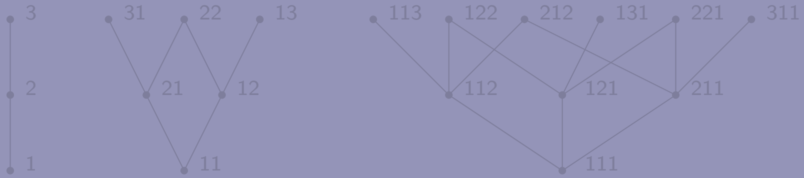
Take $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ generic linear forms such that

$$\forall\, j_1, \ldots, j_n \in \overline{1, d}, \quad \ell_{1,j_1}, \ldots, \ell_{n,j_n} \text{ are linearly independent.}$$

$$H_n(d) := \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_d,d} \mid \sum_{i=1}^{i=d} s_i \leq n - d + 1\}.$$

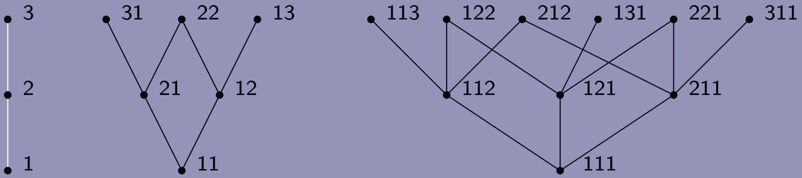
$$\ell_{s_1 1} \cdot \ldots \cdot \ell_{s_d d} \leq \ell_{t_1 1} \cdot \ldots \cdot \ell_{t_d d} \iff s_i \leq t_i, \forall\, i.$$

Here is the Hasse diagram of this poset for $d \in \{1, 2, 3\}$ and $n = 3$.



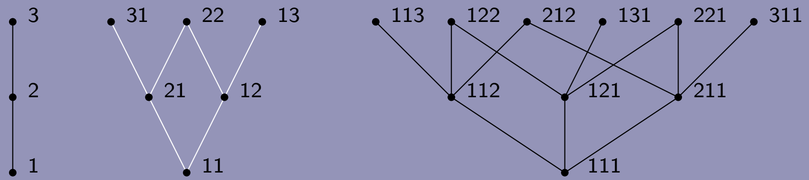
P $P^{(2)} = H_3(2)$ $P^{(3)} = H_3(3)$

Here is the Hasse diagram of this poset for $d \in \{1, 2, 3\}$ and $n = 3$.



P $\quad P^{(2)} = H_3(2) \quad P^{(3)} = H_3(3)$

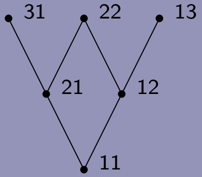
Here is the Hasse diagram of this poset for $d \in \{1, 2, 3\}$ and $n = 3$.



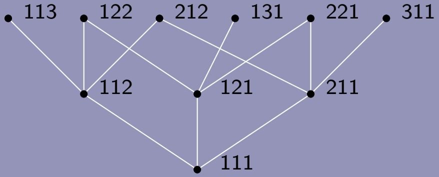
P $\quad$ $P^{(2)} = H_3(2)$ $\quad$ $P^{(3)} = H_3(3)$

Here is the Hasse diagram of this poset for $d \in \{1, 2, 3\}$ and $n = 3$.

P



$P^{(2)} = H_3(2)$



$P^{(3)} = H_3(3)$

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ as above.

$$
\begin{aligned}
H_n(j) &= \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j} \mid \sum_{i=1}^{i=j} s_i \leq n - j + 1\}. \\
\mathcal{I}(\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j}) &= \{\ell_{t_1,1} \cdot \ldots \cdot \ell_{t_d,d} \in H_n(d) \mid \\
&\qquad (t_1, \ldots, t_j, \ldots, t_d) \not\geq (s_1, \ldots, s_j, 1, \ldots, 1)\}.
\end{aligned}
$$

Theorem

For every $d \geq 2$ and for every $j \in \{0, \ldots, d-1\}$, the jth Veronese module $M_j^{(d)}$ is a homogenous MSL on $H_n(j)$ over $R^{(d)}$ with the structure defined above.

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ as above.

$$H_n(j) = \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j} \mid \sum_{i=1}^{i=j} s_i \leq n-j+1\}.$$

$$\mathcal{I}(\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j}) = \{\ell_{t_1,1} \cdot \ldots \cdot \ell_{t_d,d} \in H_n(d) \mid (t_1, \ldots, t_j, \ldots, t_d) \not\geq (s_1, \ldots, s_j, 1, \ldots, 1)\}.$$

Theorem

For every $d \geq 2$ and for every $j \in \{0, \ldots, d-1\}$, the jth Veronese module $M_j^{(d)}$ is a homogenous MSL on $H_n(j)$ over $R^{(d)}$ with the structure defined above.

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ as above.

$$\begin{aligned} H_n(j) &= \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j} \mid \sum_{i=1}^{i=j} s_i \leq n - j + 1\}. \\ \mathcal{I}(\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j}) &= \{\ell_{t_1,1} \cdot \ldots \cdot \ell_{t_d,d} \in H_n(d) \mid \\ &\quad (t_1, \ldots, t_j, \ldots, t_d) \not\geq (s_1, \ldots, s_j, 1, \ldots, 1)\}. \end{aligned}$$

Theorem

For every $d \geq 2$ and for every $j \in \{0, \ldots, d-1\}$, the jth Veronese module $M_j^{(d)}$ is a homogenous MSL on $H_n(j)$ over $R^{(d)}$ with the structure defined above.

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $\{\ell_{i,j}\}_{i=\overline{1,n},\ j=\overline{1,d}}$ as above.

$$\begin{aligned} H_n(j) &= \{\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j} \mid \sum_{i=1}^{i=j} s_i \leq n - j + 1\}. \\ \mathcal{I}(\ell_{s_1,1} \cdot \ldots \cdot \ell_{s_j,j}) &= \{\ell_{t_1,1} \cdot \ldots \cdot \ell_{t_d,d} \in H_n(d) \mid \\ &\quad (t_1, \ldots, t_j, \ldots, t_d) \not\geq (s_1, \ldots, s_j, 1, \ldots, 1)\}. \end{aligned}$$

Theorem

For every $d \geq 2$ and for every $j \in \{0, \ldots, d-1\}$, the jth Veronese module $M_j^{(d)}$ is a homogenous MSL on $H_n(j)$ over $R^{(d)}$ with the structure defined above.

By a theorem of W. Bruns, J. Herzog and U. Vetter we have as a corollary the following result, which was proved by A. Aramova, S. Bărcănescu and J. Herzog:

Corollary
For every $0 \leq j \leq d-1$ *the Veronese module* $M_j^{(d)}$ *has a linear resolution as a* $R^{(d)}$*-module.*

By a theorem of W. Bruns, J. Herzog and U. Vetter we have as a corollary the following result, which was proved by A. Aramova, S. Bărcănescu and J. Herzog:

Corollary

For every $0 \leq j \leq d-1$ the Veronese module $M_j^{(d)}$ has a linear resolution as a $R^{(d)}$-module.

By a theorem of W. Bruns, J. Herzog and U. Vetter we have as a corollary the following result, which was proved by A. Aramova, S. Bărcănescu and J. Herzog:

Corollary

For every $0 \leq j \leq d-1$ *the Veronese module* $M_j^{(d)}$ *has a linear resolution as a* $R^{(d)}$*-module.*

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

   If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

   $\operatorname{rank}(P) = \operatorname{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in } P\}.$$

   $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\text{rank}(P) = \text{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\text{rank}(P) = \text{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

STRAIGHTENING LAWS
The Combinatorial Problem

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\mathrm{rank}(P) = \mathrm{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\mathrm{rank}(P) = \mathrm{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\text{rank}(P) = \text{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

1.' If P has a unique minimal element, so should $P^{(d)}$.

2. The Krull dimension of A is equal to the rank of P.

2.' $\mathrm{rank}(P) = \mathrm{rank}(P^{(d)})$.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \#\{\text{multichains of length } i \text{ in P}\}.$$

3.' $\#\ \{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\ \}$ for all $m \geq 1$.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \dots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_1, \dots, \alpha_d) \mid \alpha_j \in P,\ \forall\, j \text{ and } \alpha_1 \leq \dots \leq \alpha_d)\}$$

and say that:

$$(\alpha_1, \dots, \alpha_d) \leq (\beta_1, \dots, \beta_d) \iff \begin{array}{rl} & \alpha_1 \leq \beta_1, \\ \text{and} & \alpha_2 \geq \beta_2, \\ \text{and} & \alpha_3 \leq \beta_3, \\ \text{and} & \alpha_4 \geq \beta_4, \\ & \dots\ etc. \end{array}$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \ldots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_1, \ldots, \alpha_d) \mid \alpha_j \in P, \ \forall\, j \text{ and } \alpha_1 \leq \ldots \leq \alpha_d)\}$$

and say that:

$$(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d) \iff \begin{array}{rl} & \alpha_1 \leq \beta_1, \\ \text{and} & \alpha_2 \geq \beta_2, \\ \text{and} & \alpha_3 \leq \beta_3, \\ \text{and} & \alpha_4 \geq \beta_4, \\ & \ldots \textit{etc.} \end{array}$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \ldots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_1, \ldots, \alpha_d) \mid \alpha_j \in P, \ \forall\, j \text{ and } \alpha_1 \leq \ldots \leq \alpha_d)\}$$

and say that:

$$(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d) \iff \begin{array}{ll} & \alpha_1 \leq \beta_1, \\ \text{and} & \alpha_2 \geq \beta_2, \\ \text{and} & \alpha_3 \leq \beta_3, \\ \text{and} & \alpha_4 \geq \beta_4, \\ & \ldots \textit{etc}. \end{array}$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \ldots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_1, \ldots, \alpha_d) \mid \alpha_j \in P, \ \forall\, j \text{ and } \alpha_1 \leq \ldots \leq \alpha_d)\}$$

and say that:

$$(\alpha_1, \ldots, \alpha_d) \leq (\beta_1, \ldots, \beta_d) \iff \begin{array}{rl} & \alpha_1 \leq \beta_1, \\ \text{and} & \alpha_2 \geq \beta_2, \\ \text{and} & \alpha_3 \leq \beta_3, \\ \text{and} & \alpha_4 \geq \beta_4, \\ & \ldots \textit{etc}. \end{array}$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \dots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_1, \dots, \alpha_d) \mid \alpha_j \in P,\ \forall\, j \text{ and } \alpha_1 \leq \dots \leq \alpha_d)\}$$

and say that:

$$(\alpha_1, \dots, \alpha_d) \leq (\beta_1, \dots, \beta_d) \quad \Longleftrightarrow \quad \begin{array}{rl} & \alpha_1 \leq \beta_1, \\ \text{and} & \alpha_2 \geq \beta_2, \\ \text{and} & \alpha_3 \leq \beta_3, \\ \text{and} & \alpha_4 \geq \beta_4, \\ & \dots\ etc. \end{array}$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

A construction that has properties 2.' and 3.' is the Zig-Zag poset.

Let $P = \{\alpha_1, \dots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_2(P) := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 \in P, \text{ and } \alpha_1 \leq \alpha_2\}$$

and say that:

$$(\alpha_1, \alpha_2) \leq (\beta_1, \beta_2) \iff \alpha_1 \leq \beta_1 \quad \text{and} \quad \alpha_2 \geq \beta_2$$

If P has a unique maximal element and $d = 2$ then 1.' also holds.

Let L be a distributive lattice. The Hibi ring on L over k is:

$$A_{Hibi} = \frac{k[\alpha]_{\alpha \in L}}{I_{Hibi}}, \quad \text{where}$$

$$I_{Hibi} = (\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta) \mid \forall\, \alpha \not\sim \beta).$$

Theorem

Let L be a distributive lattice and A_{Hibi} be the Hibi ring on L over k. Then $A^{(2)}_{Hibi}$ is an ASL over $Z_2(L)$ with the following straightening laws:

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[((\alpha \wedge \delta) \vee (\beta \wedge \gamma))((\alpha \vee \delta) \wedge (\beta \vee \gamma))\Big], \tag{3}$$

$\forall\, \alpha, \beta, \gamma, \delta \in L$, with $\alpha \leq \beta$, $\gamma \leq \delta$ and $\alpha\beta \not\sim \gamma\delta$.

Let L be a distributive lattice. The Hibi ring on L over k is:

$$A_{Hibi} = \frac{k[\alpha]_{\alpha \in L}}{I_{Hibi}}, \quad \text{where}$$

$$I_{Hibi} = (\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta) \mid \forall\, \alpha \not\sim \beta).$$

Theorem

Let L be a distributive lattice and A_{Hibi} be the Hibi ring on L over k. Then $A^{(2)}_{Hibi}$ is an ASL over $Z_2(L)$ with the following straightening laws:

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big], \tag{3}$$

$\forall\, \alpha, \beta, \gamma, \delta \in L$, with $\alpha \leq \beta$, $\gamma \leq \delta$ and $\alpha\beta \not\sim \gamma\delta$.

Let L be a distributive lattice. The Hibi ring on L over k is:

$$\begin{aligned} A_{Hibi} &= \frac{k[\alpha]_{\alpha \in L}}{I_{Hibi}}, \quad \text{where} \\ I_{Hibi} &= (\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta) \mid \forall\, \alpha \not\sim \beta). \end{aligned}$$

Theorem

Let L be a distributive lattice and A_{Hibi} be the Hibi ring on L over k. Then $A^{(2)}_{Hibi}$ is an ASL over $Z_2(L)$ with the following straightening laws:

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big], \tag{3}$$

$\forall\ \alpha, \beta, \gamma, \delta \in L,$ with $\alpha \leq \beta,\ \gamma \leq \delta$ and $\alpha\beta \not\sim \gamma\delta$.

Let L be a distributive lattice. The Hibi ring on L over k is:

$$A_{Hibi} = \frac{k[\alpha]_{\alpha \in L}}{I_{Hibi}}, \quad \text{where}$$

$$I_{Hibi} = (\alpha\beta - (\alpha \wedge \beta)(\alpha \vee \beta) \mid \forall\, \alpha \not\sim \beta).$$

Theorem

Let L be a distributive lattice and A_{Hibi} be the Hibi ring on L over k. Then $A^{(2)}_{Hibi}$ is an ASL over $Z_2(L)$ with the following straightening laws:

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big], \tag{3}$$

$\forall\ \alpha, \beta, \gamma, \delta \in L,$ with $\alpha \leq \beta,\ \gamma \leq \delta$ and $\alpha\beta \not\sim \gamma\delta.$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

STRAIGHTENING LAWS
The Hibi Ring

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha\wedge\gamma)(\beta\vee\delta)\Big]\Big[\big((\alpha\wedge\delta)\vee(\beta\wedge\gamma)\big)\big((\alpha\vee\delta)\wedge(\beta\vee\gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha,\beta,\gamma,\delta\} = \{\alpha_0,\beta_0,\gamma_0,\delta_0\}$ as multi-sets.

If $\alpha\vee\gamma$ and $\beta\wedge\delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha\wedge\gamma)(\beta\vee\delta)\big]\big[(\alpha\vee\gamma)(\beta\wedge\delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha\wedge\gamma)(\beta\vee\delta)\Big]\Big[\big((\alpha\wedge\delta)\vee(\beta\wedge\gamma)\big)\big((\alpha\vee\delta)\wedge(\beta\vee\gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha,\beta,\gamma,\delta\} = \{\alpha_0,\beta_0,\gamma_0,\delta_0\}$ as multi-sets.

If $\alpha\vee\gamma$ and $\beta\wedge\delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha\wedge\gamma)(\beta\vee\delta)\big]\big[(\alpha\vee\gamma)(\beta\wedge\delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha\wedge\gamma)(\beta\vee\delta)\Big]\Big[\big((\alpha\wedge\delta)\vee(\beta\wedge\gamma)\big)\big((\alpha\vee\delta)\wedge(\beta\vee\gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha,\beta,\gamma,\delta\} = \{\alpha_0,\beta_0,\gamma_0,\delta_0\}$ as multi-sets.

If $\alpha\vee\gamma$ and $\beta\wedge\delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha\wedge\gamma)(\beta\vee\delta)\big]\big[(\alpha\vee\gamma)(\beta\wedge\delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha\wedge\gamma)(\beta\vee\delta)\Big]\Big[\big((\alpha\wedge\delta)\vee(\beta\wedge\gamma)\big)\big((\alpha\vee\delta)\wedge(\beta\vee\gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha,\beta,\gamma,\delta\} = \{\alpha_0,\beta_0,\gamma_0,\delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha\wedge\gamma)(\beta\vee\delta)\big]\big[(\alpha\vee\gamma)(\beta\wedge\delta)\big].$$

$$(\alpha\beta)(\gamma\delta) = \Big[(\alpha \wedge \gamma)(\beta \vee \delta)\Big]\Big[\big((\alpha \wedge \delta) \vee (\beta \wedge \gamma)\big)\big((\alpha \vee \delta) \wedge (\beta \vee \gamma)\big)\Big]$$

Example

If $\{\alpha,\beta,\gamma,\delta\}$ is totally ordered, but $(\alpha\beta) \not\leq (\gamma\delta)$, then (3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets.

If $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (3) is actually:

$$(\alpha\beta)(\gamma\delta) = \big[(\alpha \wedge \gamma)(\beta \vee \delta)\big]\big[(\alpha \vee \gamma)(\beta \wedge \delta)\big].$$

- Characterize the Hilbert function of algebras with the Lefschetz property m times.
- Give upper bounds for the Betti numbers of Artinian algebras with a given Hilbert Function and with the Lefschetz property m times and describe the cases in which these bounds are reached.

- Characterize the Hilbert function of algebras with the Lefschetz property m times.
- Give upper bounds for the Betti numbers of Artinian algebras with a given Hilbert Function and with the Lefschetz property m times and describe the cases in which these bounds are reached.

- Characterize the Hilbert function of algebras with the Lefschetz property m times.
- Give upper bounds for the Betti numbers of Artinian algebras with a given Hilbert Function and with the Lefschetz property m times and describe the cases in which these bounds are reached.

- Characterize the Hilbert function of algebras with the Lefschetz property m times.

- Give upper bounds for the Betti numbers of Artinian algebras with a given Hilbert Function and with the Lefschetz property m times and describe the cases in which these bounds are reached.

T. Harima, J.C. Migliore, U. Nagel, J. Watanabe
The Weak and Strong Lefschetz Properties for Artinian K-algebras,
Journal of Algebra **262** (2003), 99–126.

T. Harima, A. Wachi
Generic initial ideals, graded Betti numbers and k-Lefschetz properties,
To appear in Communications in Algebra.

T. Harima, J.C. Migliore, U. Nagel, J. Watanabe
The Weak and Strong Lefschetz Properties for Artinian K-algebras,
Journal of Algebra **262** (2003), 99–126.

T. Harima, A. Wachi
Generic initial ideals, graded Betti numbers and k-Lefschetz properties,
To appear in Communications in Algebra.

T. Harima, J.C. Migliore, U. Nagel, J. Watanabe
The Weak and Strong Lefschetz Properties for Artinian K-algebras,
Journal of Algebra **262** (2003), 99–126.

T. Harima, A. Wachi
Generic initial ideals, graded Betti numbers and k-Lefschetz properties,
To appear in Communications in Algebra.

Let k be an infinite field, $char(k) = 0$.

$A = \bigoplus_{d \geq 0} A_d$ be a homogeneous, Artinian k-algebra, i.e.

$A = R/I$, where $R = k[x_1, \ldots, x_n]$ and I is a homogeneous ideal.

h_A - the Hilbert function of A

$h_A(d) = dim_k(A_d)$.

Let k be an infinite field, $char(k) = 0$.

$A = \bigoplus_{d \geq 0} A_d$ be a homogeneous, Artinian k-algebra, i.e.

$A = R/I$, where $R = k[x_1, \ldots, x_n]$ and I is a homogeneous ideal.

h_A - the Hilbert function of A

$h_A(d) = \dim_k(A_d)$.

Let k be an infinite field, $char(k) = 0$.

$A = \bigoplus_{d \geq 0} A_d$ be a homogeneous, Artinian k-algebra, i.e.

$A = R/I$, where $R = k[x_1, \ldots, x_n]$ and I is a homogeneous ideal.

h_A - the Hilbert function of A

$h_A(d) = \dim_k(A_d)$.

Let k be an infinite field, $char(k) = 0$.

$A = \bigoplus_{d \geq 0} A_d$ be a homogeneous, Artinian k-algebra, i.e.

$A = R/I$, where $R = k[x_1, \ldots, x_n]$ and I is a homogeneous ideal.

h_A - the Hilbert function of A

$h_A(d) = \dim_k(A_d)$.

Let k be an infinite field, $char(k) = 0$.

$A = \bigoplus_{d \geq 0} A_d$ be a homogeneous, Artinian k-algebra, i.e.

$A = R/I$, where $R = k[x_1, \dots, x_n]$ and I is a homogeneous ideal.

h_A - the Hilbert function of A

$h_A(d) = \dim_k(A_d)$.

Definition

A has the weak Lefschetz Property (WLP) if $\exists\, \ell \in A_1$ such that

$$\times \ell \ : \ A_d \longrightarrow A_{d+1}$$

has maximal rank $\forall\, d \geq 1$.

ℓ is called a weak Lefschetz Element (WLE) for A.

A has m-times the WLP ($m \in \mathbb{N}$) if $\exists\, \ell_1, \ldots, \ell_m \in A_1$ such that

$$\ell_i \text{ is a WLE for } A/(\ell_1, \ldots, \ell_{i-1}),$$

$\forall\, i \in 1, \ldots, m$.

Definition

A has the weak Lefschetz Property (WLP) if $\exists\ \ell \in A_1$ such that

$$\times \ell \ :\ A_d \longrightarrow A_{d+1}$$

has maximal rank $\forall\ d \geq 1$.

ℓ is called a weak Lefschetz Element (WLE) for A.

A has m-times the WLP ($m \in \mathbb{N}$) if $\exists\ \ell_1, \ldots, \ell_m \in A_1$ such that

$$\ell_i \text{ is a WLE for } A/(\ell_1, \ldots, \ell_{i-1}),$$

$\forall\ i \in 1, \ldots, m$.

Definition

A has the weak Lefschetz Property (WLP) if $\exists\ \ell \in A_1$ such that

$$\times\ell\ :\ A_d \longrightarrow A_{d+1}$$

has maximal rank $\forall\ d \geq 1$.

ℓ is called a weak Lefschetz Element (WLE) for A.

A has *m*-times the WLP ($m \in \mathbb{N}$) if $\exists\ \ell_1, \ \dots\ , \ \ell_m \in A_1$ such that

$$\ell_i \text{ is a WLE for } A/(\ell_1, \dots, \ell_{i-1}),$$

$\forall\ i \in 1, \dots, m$.

Definition
A has the weak Lefschetz Property (WLP) if $\exists\, \ell \in A_1$ such that

$$\times\ell \; : \; A_d \longrightarrow A_{d+1}$$

has maximal rank $\forall\, d \geq 1$.

ℓ is called a weak Lefschetz Element (WLE) for A.

A has m-times the WLP ($m \in \mathbb{N}$) if $\exists\, \ell_1, \ldots, \ell_m \in A_1$ such that

$$\ell_i \text{ is a WLE for } A/(\ell_1, \ldots, \ell_{i-1}),$$

$\forall\, i \in 1, \ldots, m$.

Definition

A has the weak Lefschetz Property (WLP) if $\exists\, \ell \in A_1$ such that

$$\times \ell \;:\; A_d \longrightarrow A_{d+1}$$

has maximal rank $\forall\, d \geq 1$.

ℓ is called a weak Lefschetz Element (WLE) for A.

A has *m*-times the WLP ($m \in \mathbb{N}$) if $\exists\, \ell_1, \ldots, \ell_m \in A_1$ such that

$$\ell_i \text{ is a WLE for } A/(\ell_1, \ldots, \ell_{i-1}),$$

$\forall\, i \in 1, \ldots, m$.

Let $h \ : \ 1 = h_0, \ h_1, \ \ldots, \ h_s$ be a finite O-sequence.

Definition

h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 \le h_1 \le \ldots \le h_k \ge h_{k+1} \ge \ldots \ge h_s$ for some $k \in 0, \ldots, s$.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is again an O-sequence.

h is a m-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Let $h\ :\ 1 = h_0,\ h_1,\ \ldots,\ h_s$ be a finite O-sequence.

Definition

h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ for some $k \in 0, \ldots, s$.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is again an O-sequence.

h is a m-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Let $h : 1 = h_0, h_1, \ldots, h_s$ be a finite O-sequence.

Definition

h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ for some $k \in 0, \ldots, s$.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is again an O-sequence.

h is a m-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Let $h\ :\ 1 = h_0,\ h_1,\ \ldots,\ h_s$ be a finite O-sequence.

Definition

h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ for some $k \in 0, \ldots, s$.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is again an O-sequence.

h is a m-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Let $h\ :\ 1 = h_0,\ h_1,\ \dots,\ h_s$ be a finite O-sequence.

Definition
h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ for some $k \in 0, \ldots, s$.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is again an O-sequence.

h is a m-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Let $h\ :\ 1 = h_0,\ h_1,\ \dots,\ h_s$ be a finite O-sequence.

Definition

h is a weak Lefschetz O-sequence if :

- h is unimodal i.e. $h_0 < h_1 < \dots < h_k \geq h_{k+1} \geq \dots \geq h_s$ for some $k \in 0, \dots, s$.
- the sequence $1, h_1 - h_0, \dots, h_k - h_{k-1}$ is again an O-sequence.

h is a *m*-times weak Lefschetz O-sequence if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \dots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

Theorem

1. *If A is an Artinian homogeneous k-algebra with m-times the WLP, then h_A is a m-times weak Lefschetz O-sequence.*
2. *For every m-times weak Lefschetz O-sequence h, there exists an Artinian homogeneous k-algebra with $h_A = h$.*

1. is easy:

- The unimodality follows from the natural grading of the algebra:
  if $\times \ell_1 : A_j \longrightarrow A_{j+1}$ is surjective, then $\times \ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$.
- $A/(\ell_1)$ in an algebra with $(m-1)$-times the WLP and

$$h_{A/(\ell_1)} : 1, h_1 - h_0, \ldots, h_k - h_{k-1}.$$

Theorem

1. *If A is an Artinian homogeneous k-algebra with m-times the WLP, then h_A is a m-times weak Lefschetz O-sequence.*
2. *For every m-times weak Lefschetz O-sequence h, there exists an Artinian homogeneous k-algebra with $h_A = h$.*

1. is easy:

- The unimodality follows from the natural grading of the algebra:
  if $\times\ell_1 : A_j \longrightarrow A_{j+1}$ is surjective, then $\times\ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$.
- $A/(\ell_1)$ in an algebra with $(m-1)$-times the WLP and

$$h_{A/(\ell_1)} : 1, h_1 - h_0, \ldots, h_k - h_{k-1}.$$

Theorem

1. *If A is an Artinian homogeneous k-algebra with m-times the WLP, then h_A is a m-times weak Lefschetz O-sequence.*
2. *For every m-times weak Lefschetz O-sequence h, there exists an Artinian homogeneous k-algebra with $h_A = h$.*

1. is easy:

- The unimodality follows from the natural grading of the algebra:
  if $\times\ell_1 : A_j \longrightarrow A_{j+1}$ is surjective, then $\times\ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$.
- $A/(\ell_1)$ in an algebra with $(m-1)$-times the WLP and

$$h_{A/(\ell_1)} : 1, h_1 - h_0, \ldots, h_k - h_{k-1}.$$

Theorem

1. *If A is an Artinian homogeneous k-algebra with m-times the WLP, then h_A is a m-times weak Lefschetz O-sequence.*
2. *For every m-times weak Lefschetz O-sequence h, there exists an Artinian homogeneous k-algebra with $h_A = h$.*

1. is easy:

- The unimodality follows from the natural grading of the algebra:
  if $\times\ell_1 : A_j \longrightarrow A_{j+1}$ is surjective, then $\times\ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$.
- $A/(\ell_1)$ in an algebra with $(m-1)$-times the WLP and

$$h_{A/(\ell_1)} : 1, h_1 - h_0, \dots, h_k - h_{k-1}.$$

Theorem

1. *If A is an Artinian homogeneous k-algebra with m-times the WLP, then h_A is a m-times weak Lefschetz O-sequence.*
2. *For every m-times weak Lefschetz O-sequence h, there exists an Artinian homogeneous k-algebra with $h_A = h$.*

1. is easy:

- The unimodality follows from the natural grading of the algebra:
  if $\times\ell_1 : A_j \longrightarrow A_{j+1}$ is surjective, then $\times\ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$.
- $A/(\ell_1)$ in an algebra with $(m-1)$-times the WLP and

$$h_{\mathrm{A}/(\ell_1)} : 1, h_1 - h_0, \ldots, h_k - h_{k-1}.$$

2. is not so easy....

Fix h a m-times weak Lefschetz O-sequence.

1. We construct inductively an ideal $\mathcal{W}_m(h)$ of R such that $R/\mathcal{W}_m(h)$ will be the algebra we are looking for.

2. If R/I is an Artinian k-algebra with Hilbert function h and m-times the WLP then:

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathcal{W}_m(h)), \qquad \forall i,j \geq 0. \qquad (4)$$

Fix h a m-times weak Lefschetz O-sequence.

1. We construct inductively an ideal $\mathcal{W}_m(h)$ of R such that $R/\mathcal{W}_m(h)$ will be the algebra we are looking for.

2. If R/I is an Artinian k-algebra with Hilbert function h and m-times the WLP then:

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathcal{W}_m(h)), \qquad \forall i,j \geq 0. \tag{4}$$

Fix h a m-times weak Lefschetz O-sequence.

1. We construct inductively an ideal $\mathcal{W}_m(h)$ of R such that $R/\mathcal{W}_m(h)$ will be the algebra we are looking for.
2. If R/I is an Artinian k-algebra with Hilbert function h and m-times the WLP then:

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathcal{W}_m(h)), \qquad \forall i,j \geq 0. \tag{4}$$

Fix h a m-times weak Lefschetz O-sequence.

1. We construct inductively an ideal $\mathcal{W}_m(h)$ of R such that $R/\mathcal{W}_m(h)$ will be the algebra we are looking for.

2. If R/I is an Artinian k-algebra with Hilbert function h and m-times the WLP then:

$$\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathcal{W}_m(h)), \qquad \forall i,j \geq 0. \tag{4}$$

3. Let $I \subset R$ be an ideal such that R/I has Hilbert function h and m-times the weak Lefschetz property ($m \in \mathbb{N}$). T.F.A.E.:
   (a) R/I has maximal Betti numbers among k-algebras with the above properties.
   (b) I is componentwise linear and the ideal $\rho_{n-m}(\mathrm{Gin}(I))$ is Gotzmann in $k[x_1, \ldots, x_{n-m}]$.

where: $\rho_i : k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_i]$, with:

$$\rho_i(x_j) = \begin{cases} x_j & \text{if } j \leq i \\ 0 & \text{if } j > i. \end{cases}$$

3. Let $I \subset R$ be an ideal such that R/I has Hilbert function h and m-times the weak Lefschetz property ($m \in \mathbb{N}$). T.F.A.E.:
   (a) R/I has maximal Betti numbers among k-algebras with the above properties.
   (b) I is componentwise linear and the ideal $\rho_{n-m}(\mathrm{Gin}(I))$ is Gotzmann in $k[x_1, \ldots, x_{n-m}]$.

where: $\rho_i : k[x_1, \ldots, x_n] \longrightarrow k[x_1, \ldots, x_i]$, with:

$$\rho_i(x_j) = \begin{cases} x_j & \text{if } j \leq i \\ 0 & \text{if } j > i. \end{cases}$$

4. Let R/I be as above. If $\exists\, q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\mathbb{X}}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\mathbb{X}}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :

   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\mathbb{X}}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\leq k}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\leq k}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)),\ \text{for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\leq k}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

4. Let R/I be as above. If $\exists\ q \in \mathbb{N}$ such that:

$$\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$$

then:

$$\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h)), \text{ for all } i \geq q.$$

5. Construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I with :
   - the same Hilbert function
   - the same Betti numbers
   - such that R/I still has m-times the WLP
   - $I_{\leq k}$ is the ideal of finite set of rational points in $\mathbb{P}_k^{n-1}$.

Fix $h: 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$I_0 := \text{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}].$$
$$I_1 := I_0 \cdot R \subset R.$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

Fix $h : 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$\begin{aligned} I_0 &:= \mathrm{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}]. \\ I_1 &:= I_0 \cdot R \subset R. \end{aligned}$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots.$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

LEFSCHETZ PROPERTY

Sketch of the proof

Fix $h : 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$\begin{aligned} I_0 &:= \operatorname{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}]. \\ I_1 &:= I_0 \cdot R \subset R. \end{aligned}$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ \ldots,\ h_k,\ \ \ldots.$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

Fix $h : 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$
\begin{aligned}
I_0 &:= \mathrm{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}]. \\
I_1 &:= I_0 \cdot R \subset R.
\end{aligned}
$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots.$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

Fix $h: 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$\begin{aligned} I_0 &:= \mathrm{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}]. \\ I_1 &:= I_0 \cdot R \subset R. \end{aligned}$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots.$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

Fix $h : 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a WLO.
Denote by $\Delta h := 1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$.

Set $n = h_1$ and define:

$$\begin{aligned} I_0 &:= \operatorname{Lex}(\Delta h) \subset R' = k[x_1, \ldots, x_{n-1}]. \\ I_1 &:= I_0 \cdot R \subset R. \end{aligned}$$

It is easy to see that:

- the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots.$$

- $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \dots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \dots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \dots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \dots,\ h_{d_0},\ \dots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = \mathcal{W}_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \ldots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \ldots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = W_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \dots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \dots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \dots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \dots,\ h_{d_0},\ \dots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_1 = \mathbb{W}_{m-1}(I) \subset R'$ and follow the same steps as in the case $m = 1$.

LEFSCHETZ PROPERTY
Sketch of the proof

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \ldots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \ldots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = W_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \ldots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \ldots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = \mathcal{W}_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \ldots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \ldots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = \mathcal{W}_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

Let $d_0 > k$ be the smallest degree for which $h_k > h_{d_0}$ and denote by $r_0 := h_k - h_{d_0}$.

Take $M_1, \ldots, M_{r_0} \in R$, the largest in rev-lex order r_0 monomials of degree d_0 NOT in I_1.
Define:

$$I_2 := I_1 + (M_1, \ldots, M_{r_0}).$$

The Hilbert function of R/I_2 will be:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Proceed in the same way with I_2, and so on.
In the general case start with $I_0 = \mathcal{W}_{m-1}(h) \subset R'$ and follow the same steps as in the case $m = 1$.

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.

$R = k[x, y, z, t]$.

$$\begin{aligned} I_0 &:= \operatorname{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\ I_1 &:= I_0 \cdot R. \end{aligned}$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$$\begin{array}{lcllllllll} d = 4 & : & z^3t & y^2t^2 & yzt^2 & z^2t^2 & xt^3 & yt^3 & zt^3 & t^4 \\ d = 5 & : & z^3t^2 & y^2t^3 & yzt^3 & z^2t^3 & xt^4 & yt^4 & zt^4 & t^5 \\ d = 6 & : & z^3t^3 & y^2t^4 & yzt^4 & z^2t^4 & xt^5 & yt^5 & zt^5 & t^6 \\ d = 7 & : & z^3t^4 & y^2t^5 & yzt^5 & z^2t^5 & xt^6 & yt^6 & zt^6 & t^7 \end{array}$$

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.

$R = k[x, y, z, t]$.

$$\begin{aligned} I_0 &:= \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\ I_1 &:= I_0 \cdot R. \end{aligned}$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$$\begin{array}{lcllllllll} d = 4 & : & z^3t & y^2t^2 & yzt^2 & z^2t^2 & xt^3 & yt^3 & zt^3 & t^4 \\ d = 5 & : & z^3t^2 & y^2t^3 & yzt^3 & z^2t^3 & xt^4 & yt^4 & zt^4 & t^5 \\ d = 6 & : & z^3t^3 & y^2t^4 & yzt^4 & z^2t^4 & xt^5 & yt^5 & zt^5 & t^6 \\ d = 7 & : & z^3t^4 & y^2t^5 & yzt^5 & z^2t^5 & xt^6 & yt^6 & zt^6 & t^7 \end{array}$$

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.

$R = k[x, y, z, t]$.

$$I_0 := \text{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

LEFSCHETZ PROPERTY

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$I_0 := \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

LEFSCHETZ PROPERTY

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. $\quad \Delta h$: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$I_0 := \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

L E F S C H E T Z   P R O P E R T Y

E x a m p l e

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$\begin{aligned} I_0 &:= \operatorname{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\ I_1 &:= I_0 \cdot R. \end{aligned}$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$\begin{aligned} I_0 &:= \operatorname{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\ I_1 &:= I_0 \cdot R. \end{aligned}$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$$\begin{array}{lcllllllll} d = 4 & : & z^3t & y^2t^2 & yzt^2 & z^2t^2 & xt^3 & yt^3 & zt^3 & t^4 \\ d = 5 & : & z^3t^2 & y^2t^3 & yzt^3 & z^2t^3 & xt^4 & yt^4 & zt^4 & t^5 \\ d = 6 & : & z^3t^3 & y^2t^4 & yzt^4 & z^2t^4 & xt^5 & yt^5 & zt^5 & t^6 \\ d = 7 & : & z^3t^4 & y^2t^5 & yzt^5 & z^2t^5 & xt^6 & yt^6 & zt^6 & t^7 \end{array}$$

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$I_0 := \text{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$I_0 := \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

LEFSCHETZ PROPERTY

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.

$R = k[x, y, z, t]$.

$$I_0 := \text{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$\begin{aligned} I_0 &:= \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\ I_1 &:= I_0 \cdot R. \end{aligned}$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$$\begin{array}{lcllllllll} d = 4 & : & z^3t & y^2t^2 & yzt^2 & z^2t^2 & xt^3 & yt^3 & zt^3 & t^4 \\ d = 5 & : & z^3t^2 & y^2t^3 & yzt^3 & z^2t^3 & xt^4 & yt^4 & zt^4 & t^5 \\ d = 6 & : & z^3t^3 & y^2t^4 & yzt^4 & z^2t^4 & xt^5 & yt^5 & zt^5 & t^6 \\ d = 7 & : & z^3t^4 & y^2t^5 & yzt^5 & z^2t^5 & xt^6 & yt^6 & zt^6 & t^7 \end{array}$$

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$I_0 := \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z].$$
$$I_1 := I_0 \cdot R.$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$	:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$	:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$	:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$	:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

LEFSCHETZ PROPERTY

Example

h: 1 , 4 , 7 , 8 , 6 , 3 , 1 , 0. Δh: 1 , 3 , 3 , 1.
$R = k[x, y, z, t]$.

$$
\begin{aligned}
I_0 &:= \mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset k[x, y, z]. \\
I_1 &:= I_0 \cdot R.
\end{aligned}
$$

For every $d \geq 4$, the monomials of degree d in $R \setminus I_1$ are:

$$z^3t^{d-3},\ y^2t^{d-2},\ yzt^{d-2},\ z^2t^{d-2},\ xt^{d-1},\ yt^{d-1},\ zt^{d-1},\ t^d$$

Add to I_1:

$d = 4$:	z^3t	y^2t^2	yzt^2	z^2t^2	xt^3	yt^3	zt^3	t^4
$d = 5$:	z^3t^2	y^2t^3	yzt^3	z^2t^3	xt^4	yt^4	zt^4	t^5
$d = 6$:	z^3t^3	y^2t^4	yzt^4	z^2t^4	xt^5	yt^5	zt^5	t^6
$d = 7$:	z^3t^4	y^2t^5	yzt^5	z^2t^5	xt^6	yt^6	zt^6	t^7

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \ldots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \ldots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \ldots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \ldots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \ldots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \ldots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \ldots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \ldots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

PARAMETRIZATIONS
Introduction

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \ldots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \ldots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.

If we have that $\dim_k(k[x_1, \ldots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \ldots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.

Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

PARAMETRIZATIONS

Introduction

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \ldots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \ldots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \ldots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \ldots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \dots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \dots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \dots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \dots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \dots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \dots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \dots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \dots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \ldots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \ldots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \ldots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \ldots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

For a field k of any characteristic, a monomial ideal $I_0 \subset k[x_1, \dots, x_n]$ and any term order τ, the set

$$V_h(I_0) = \{I \subset k[x_1, \dots, x_n] \mid I \text{ homogeneous, with } \mathrm{in}_\tau(I) = I_0\}$$

has a natural structure of affine variety.
If we have that $\dim_k(k[x_1, \dots, x_n]/I_0) < \infty$, also

$$V(I_0) := \{I \subset k[x_1, \dots, x_n] \mid \mathrm{in}_\tau(I) = I_0\}$$

has a structure of affine variety.
Main goal: parametrize the affine variety $V(I_0)$, when:

- I_0 is a monomial, lex-segment ideal of $R = k[x, y]$,
- τ is the degree reverse-lexicographic (DRL) term order,
- and $\dim_k(R/I_0) < \infty$.

Some results in this direction

J. Briançon and A. Iarrobino proved (independently) that in the above situation $V(I_0)$ is an affine space.

A. Bialynicki-Birula proves some general results which imply that $V(I_0)$ is an affine space.

Already in the case of three variables this is no longer true without extra assumptions.

A. Conca and G. Valla parametrized $V(I_0)$ when τ is the lexicographic term order.

J. Briançon and A. Iarrobino proved (independently) that in the above situation $V(I_0)$ is an affine space.

A. Bialynicki-Birula proves some general results which imply that $V(I_0)$ is an affine space.

Already in the case of three variables this is no longer true without extra assumptions.

A. Conca and G. Valla parametrized $V(I_0)$ when τ is the lexicographic term order.

J. Briançon and A. Iarrobino proved (independently) that in the above situation $V(I_0)$ is an affine space.

A. Bialynicki-Birula proves some general results which imply that $V(I_0)$ is an affine space.

Already in the case of three variables this is no longer true without extra assumptions.

A. Conca and G. Valla parametrized $V(I_0)$ when τ is the lexicographic term order.

J. Briançon and A. Iarrobino proved (independently) that in the above situation $V(I_0)$ is an affine space.

A. Bialynicki-Birula proves some general results which imply that $V(I_0)$ is an affine space.

Already in the case of three variables this is no longer true without extra assumptions.

A. Conca and G. Valla parametrized $V(I_0)$ when τ is the lexicographic term order.

J. Briançon and A. Iarrobino proved (independently) that in the above situation $V(I_0)$ is an affine space.

A. Bialynicki-Birula proves some general results which imply that $V(I_0)$ is an affine space.

Already in the case of three variables this is no longer true without extra assumptions.

A. Conca and G. Valla parametrized $V(I_0)$ when τ is the lexicographic term order.

The parametrization that we will find associates to each ideal $I \in V(I_0)$ a canonical Hilbert-Burch matrix. That is we associate to an ideal I a matrix such that:

- The maximal minors generate the ideal I.
- The columns generate the syzygy module $Syz(I)$.

This will also allow us to find a formula for the dimension of $V(I_0)$.

The parametrization that we will find associates to each ideal $I \in V(I_0)$ a canonical Hilbert-Burch matrix. That is we associate to an ideal I a matrix such that:

- The maximal minors generate the ideal I.
- The columns generate the syzygy module $Syz(I)$.

This will also allow us to find a formula for the dimension of $V(I_0)$.

The parametrization that we will find associates to each ideal $I \in V(I_0)$ a canonical Hilbert-Burch matrix. That is we associate to an ideal I a matrix such that:

- The maximal minors generate the ideal I.
- The columns generate the syzygy module $Syz(I)$.

This will also allow us to find a formula for the dimension of $V(I_0)$.

The parametrization that we will find associates to each ideal $I \in V(I_0)$ a canonical Hilbert-Burch matrix. That is we associate to an ideal I a matrix such that:

- The maximal minors generate the ideal I.
- The columns generate the syzygy module $Syz(I)$.

This will also allow us to find a formula for the dimension of $V(I_0)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x,y,z]$, with:

- $J_0 = I_0 k[x,y,z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x,y]$.

Let $S = k[x,y,z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \text{in}(I) = \text{Lex}(h)S\}.$$

When $\text{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x,y,z]$, with:

- $J_0 = I_0 k[x,y,z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x,y]$.

Let $S = k[x,y,z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \text{in}(I) = \text{Lex}(h)S\}.$$

When $\text{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x, y, z]$, with:

- $J_0 = I_0 k[x, y, z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x, y]$.

Let $S = k[x, y, z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \mathrm{in}(I) = \mathrm{Lex}(h)S\}.$$

When $\mathrm{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x,y,z]$, with:

- $J_0 = I_0 k[x,y,z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x,y]$.

Let $S = k[x,y,z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \mathrm{in}(I) = \mathrm{Lex}(h)S\}.$$

When $\mathrm{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x,y,z]$, with:

- $J_0 = I_0 k[x,y,z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x,y]$.

Let $S = k[x,y,z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \mathrm{in}(I) = \mathrm{Lex}(h)S\}.$$

When $\mathrm{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x,y,z]$, with:

- $J_0 = I_0 k[x,y,z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x,y]$.

Let $S = k[x,y,z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \mathrm{in}(I) = \mathrm{Lex}(h)S\}.$$

When char$(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

We will also parametrize $V_h(J_0)$ when $J_0 \subset k[x, y, z]$, with:

- $J_0 = I_0 k[x, y, z]$ and
- I_0 is a monomial, lex-segment ideal of $k[x, y]$.

Let $S = k[x, y, z]$, fix $H(t) = \frac{h(t)}{1-t}$ a Hilbert series and denote by

$$\mathbb{G}(H) = \{I \subset S \mid H_{S/I} = H, I \text{ is an ideal of points of } \mathbb{P}^2\}.$$

In characteristic 0, this parametrization will allow us to study the Betti strata of the affine set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \mathrm{in}(I) = \mathrm{Lex}(h)S\}.$$

When $\mathrm{char}(k) = 0$, the set $\mathbb{G}^*_{Lex}(H)$ is a Zariski open subset of the affine variety $\mathbb{G}(H)$.

Let $I_0 \subset R = k[x,y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \dots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \dots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \dots & 0 \\ -x & y^{d_2} & \dots & 0 \\ 0 & -x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{d_t} \\ 0 & 0 & \dots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

Let $I_0 \subset R = k[x, y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \ldots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \ldots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

PARAMETRIZATIONS

Ideals of $k[x, y]$

Let $I_0 \subset R = k[x, y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \ldots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \ldots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

Let $I_0 \subset R = k[x,y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \dots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \dots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \dots & 0 \\ -x & y^{d_2} & \dots & 0 \\ 0 & -x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{d_t} \\ 0 & 0 & \dots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

PARAMETRIZATIONS

Ideals of $k[x,y]$

Let $I_0 \subset R = k[x,y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \dots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \dots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \dots & 0 \\ -x & y^{d_2} & \dots & 0 \\ 0 & -x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{d_t} \\ 0 & 0 & \dots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

Let $I_0 \subset R = k[x,y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \dots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \dots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \dots & 0 \\ -x & y^{d_2} & \dots & 0 \\ 0 & -x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{d_t} \\ 0 & 0 & \dots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

Let $I_0 \subset R = k[x,y]$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that $0 = m_0 \leq m_1 \leq \ldots \leq m_t$.
Define $d_i := m_i - m_{i-1}$ for all $i \in \{1, \ldots, t\}$.

Now define the following $(t+1) \times t$ matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

Notice that the matrix X is a Hilbert-Burch matrix for I_0.

PARAMETRIZATIONS

Ideals of $k[x, y]$

Let A be another $(t+1) \times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \leq \begin{cases} \mathrm{Min}\{i - j + m_j - m_{i-1} - 1, \quad d_i - 1\} & \text{if } i \leq j, \\ \mathrm{Min}\{i - j + m_j - m_{i-1} \quad\quad, \quad d_j - 1\} & \text{if } i > j. \end{cases}$$

We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition.

Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$.

Let A be another $(t+1) \times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \leq \begin{cases} \text{Min}\{i-j+m_j-m_{i-1}-1, \quad d_i-1\} & \text{if } i \leq j, \\ \text{Min}\{i-j+m_j-m_{i-1} \quad\quad, \quad d_j-1\} & \text{if } i > j. \end{cases}$$

We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition.

Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$.

Let A be another $(t+1) \times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \leq \begin{cases} \text{Min}\{i-j+m_j-m_{i-1}-1, \quad d_i-1\} & \text{if } i \leq j, \\ \text{Min}\{i-j+m_j-m_{i-1} \qquad, \quad d_j-1\} & \text{if } i > j. \end{cases}$$

We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition.

Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$.

Let A be another $(t+1)\times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \le \begin{cases} \mathrm{Min}\{i-j+m_j-m_{i-1}-1, \quad d_i-1\} & \text{if } i\le j, \\ \mathrm{Min}\{i-j+m_j-m_{i-1} \quad , \quad d_j-1\} & \text{if } i>j. \end{cases}$$

We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition.

Notice that $\mathscr{A}_{I_0}=\mathbb{A}^N$.

Let A be another $(t+1) \times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \leq \begin{cases} \mathrm{Min}\{i - j + m_j - m_{i-1} - 1, \quad d_i - 1\} & \text{if } i \leq j, \\ \mathrm{Min}\{i - j + m_j - m_{i-1} \qquad , \quad d_j - 1\} & \text{if } i > j. \end{cases}$$

We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition.

Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$.

Let $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by:

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$.

Theorem

Let $I_0 \subset R = k[x,y]$ be a monomial lex-segment ideal with $\dim_k(R/I_0) < \infty$.
Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.

Let $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by:

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$.

Theorem

*Let $I_0 \subset R = k[x,y]$ be a monomial lex-segment ideal with $\dim_k(R/I_0) < \infty$.
Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.*

Let $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by:

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$.

Theorem

Let $I_0 \subset R = k[x, y]$ be a monomial lex-segment ideal with $\dim_k(R/I_0) < \infty$.
Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.

Let $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by:

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$.

Theorem

Let $I_0 \subset R = k[x,y]$ be a monomial lex-segment ideal with $\dim_k(R/I_0) < \infty$.
Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

So

$$
\begin{aligned}
I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\
m\text{'s} &: \quad 0,\ 2,\ 3,\ 5. \\
d\text{'s} &: \quad 2,\ 1,\ 2.
\end{aligned}
$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

So

$$
\begin{aligned}
I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\
m's &: \quad 0,\ 2,\ 3,\ 5 \\
d's &: \quad 2,\ 1,\ 2.
\end{aligned}
$$

P A R A M E T R I Z A T I O N S

E x a m p l e

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1,\\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3,\\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st Find a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5).\\
m\text{'s} &: \; 0, \; 2, \; 3, \; 5\\
d\text{'s} &: \; 2, \; 1, \; 2.
\end{aligned}
$$

PARAMETRIZATIONS

Example

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\
m\text{'s} &: \quad 0,\ 2,\ 3,\ 5. \\
d\text{'s} &: \quad 2,\ 1,\ 2.
\end{aligned}
$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\ f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\ f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\ f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y, \end{aligned}$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$\begin{aligned} I_0 &= (x^3, x^2y^2, xy^3, y^5). \\ m\text{'s} &: 0, 2, 3, 5 \\ d\text{'s} &: 2, 1, 2. \end{aligned}$$

PARAMETRIZATIONS

Example

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\ f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\ f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\ f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y, \end{aligned}$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$\begin{aligned} I_0 &= (x^3, x^2y^2, xy^3, y^5). \\ m\text{'s} &: 0,\ 2,\ 3,\ 5 \\ d\text{'s} &: 2,\ 1,\ 2. \end{aligned}$$

PARAMETRIZATIONS
Example

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1,\\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3,\\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3y^0, x^2y^2, xy^3, y^5).\\
m\text{'s} &: 0,\ 2,\ 3,\ 5.\\
d\text{'s} &: 2,\ 1,\ 2.
\end{aligned}
$$

PARAMETRIZATIONS

Example

Let $I \subset k[x,y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1,\\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3,\\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3, x^2y^2, xy^3, y^5).\\
m\text{'s} &: 0,\ 2,\ 3,\ 5.\\
d\text{'s} &: 2,\ 1,\ 2.
\end{aligned}
$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\
m\text{'s} &: \quad 0,\ 2,\ 3,\ 5. \\
d\text{'s} &: \quad 2,\ 1,\ 2.
\end{aligned}
$$

PARAMETRIZATIONS

Example

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1,\\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3,\\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3, x^2y^2, xy^3, y^5).\\
m\text{'s} &: 0,\ 2,\ 3,\ 5.\\
d\text{'s} &: 2,\ 1,\ 2.
\end{aligned}
$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\ f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\ f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\ f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y, \end{aligned}$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$\begin{aligned} I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\ m\text{'s} &: 0,\ 2,\ 3,\ 5. \\ d\text{'s} &: 2,\ 1,\ 2. \end{aligned}$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$
\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1,\\
f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3,\\
f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y,
\end{aligned}
$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$
\begin{aligned}
I_0 &= (x^3, x^2y^2, xy^3, y^5).\\
m\text{'s} &: 0,\ 2,\ 3,\ 5.\\
d\text{'s} &: 2,\ 1,\ 2.
\end{aligned}
$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\ f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\ f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\ f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y, \end{aligned}$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$\begin{aligned} I_0 &= (x^3, x^2y^2, xy^3, y^5). \\ m\text{'s} &: 0, 2, 3, 5. \\ d\text{'s} &: 2, 1, 2. \end{aligned}$$

Let $I \subset k[x, y]$ be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3, \\ f_1 &= x^2y^2 - xy^3 - y^4 - x^2 + xy + y^2 + 1, \\ f_2 &= xy^3 - y^4 + xy^2 - xy + 4y^2 - y - 3, \\ f_3 &= y^5 + x^2y^2 + 3xy^2 - y^3 - xy - 3x - y, \end{aligned}$$

1st $\{f_0, f_1, f_2, f_3\}$ is already a Gröbner basis of I. So

$$\begin{aligned} I_0 &= (x^3 \quad , x^2y^2, xy^3, y^5). \\ m\text{'s} &: \ 0,\ 2,\ 3,\ 5. \\ d\text{'s} &: \ 2,\ 1,\ 2. \end{aligned}$$

PARAMETRIZATIONS

Example

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}$, $S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x + y & y & y + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x + y \end{pmatrix}$$

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}$, $S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2-1 & 0 & y^2 \\ -x+y & y & y+3 \\ 1 & -x & y^2+1 \\ 0 & 1 & -x+y \end{pmatrix}$$

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$. We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x + y & y & y + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x + y \end{pmatrix}$$

P A R A M E T R I Z A T I O N S

E x a m p l e

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x + y & y & y + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x + y \end{pmatrix}$$

PARAMETRIZATIONS

Example

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$. We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x + y & y & y + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x + y \end{pmatrix}$$

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x \ + y & y & y \ + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x \ + y \end{pmatrix}$$

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x + y & y & y + 3 \\ 1 & -x & y^2 + 1 \\ 0 & 1 & -x + y \end{pmatrix}$$

$$X$$

2nd Check that no monomial in the support of f_i is divisible by x^d, with $d \geq t = 3$, (except for $\mathrm{in}(f_0)$). OK.

3rd Find a Hilbert-Burch matrix for I corresponding to this Gröbner basis.

To do this we divide the S-polynomials $S_{1,0}, S_{2,1}$ and $S_{3,2}$ by $\{f_0, f_1, f_2, f_3\}$.We obtain:

$$\begin{pmatrix} y^2 - 1 & 0 & y^2 \\ -x \ \ + y & y0 & y \ \ + 3 \\ 1 & -x0 & y^2 + 1 \\ 0 & 1 & -x \ \ + y \end{pmatrix}$$

$$X + A'$$

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

PARAMETRIZATIONS

Example

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A' = \begin{pmatrix} -1 & 0 & y^2 \\ y & 0 & y+3 \\ 1 & 0 & 1 \\ 0 & 1 & y \end{pmatrix}$$

To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & {\color{red}\blacksquare} \\ -x & \blacksquare & {\color{red}\blacksquare} \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \blacksquare & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & y^2 \\ -x+y & y & y+3 \\ 1 & -x & y^2+1 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \square & \blacksquare & \color{red}\blacksquare \\ -x & \blacksquare & \color{red}\blacksquare \\ \square & -x & \blacksquare \\ \square & \blacksquare & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & y^2 \\ -x+y & y & y+3 \\ 1 & -x & y^2+1 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} y^2 & \blacksquare & y^2 \\ -x & \blacksquare & \blacksquare \\ \square & -x & \blacksquare \\ \square & \blacksquare & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & y^2 \\ -x+y & y & y+3 \\ 1 & -x & y^2+1 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\square \\ -x & \blacksquare & \blacksquare x \\ \blacksquare & -x & \blacksquare\square \\ \blacksquare & \blacksquare & -x\square \end{pmatrix}$$

$$\begin{pmatrix} y^2 - 1 & 0 & 1 \\ -x + y & y & x + 3 \\ 1 & -x & y^2 \\ 0 & 1 & -x + y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x & \blacksquare & \blacksquare x \\ \blacksquare & -x & \blacksquare\blacksquare \\ \blacksquare & \blacksquare & -x\blacksquare \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y & x+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x & \blacksquare & \blacksquare x \\ \blacksquare & -x & \blacksquare\blacksquare \\ \square & \square & -x\square \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y & x+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x\square & \blacksquare\square & \blacksquare x - x\blacksquare \\ \blacksquare & -x & \blacksquare\blacksquare \\ \blacksquare & \blacksquare & -x\blacksquare \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y & x+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x\blacksquare & \blacksquare\blacksquare & \blacksquare{\color{red}\blacksquare} \\ \blacksquare & -x & \blacksquare\blacksquare \\ \blacksquare & \blacksquare & -x\blacksquare \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & y+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ -x & \blacksquare & {\color{red}\blacksquare} \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \blacksquare & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & y+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

P A R A M E T R I Z A T I O N S

E x a m p l e

$$\begin{pmatrix} \blacksquare & \square & \blacksquare \\ -x & \square & {\color{red}\blacksquare} \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \square & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2 - 1 & 0 & 1 \\ -x + y & y + 1 & y + 3 \\ 1 & -x & y^2 \\ 0 & 1 & -x + y \end{pmatrix}$$

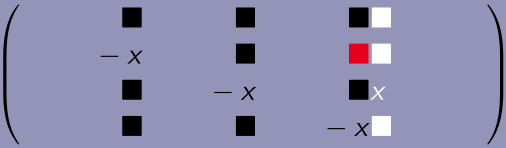
PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \square & \blacksquare \\ -x & y & y \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \square & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & y+3 \\ 1 & -x & y^2 \\ 0 & 1 & -x+y \end{pmatrix}$$

Example



$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & 2 \\ 1 & -x & y^2+x \\ 0 & 1 & -x+y-1 \end{pmatrix}$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x & \blacksquare & \blacksquare\blacksquare \\ \blacksquare & -x & \blacksquare\color{red}{x} \\ \square & \square & -x\square \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & 2 \\ 1 & -x & y^2+x \\ 0 & 1 & -x+y-1 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x & \blacksquare & \blacksquare\blacksquare \\ \blacksquare\square & -x\square & \blacksquare x - x\square \\ \blacksquare & \blacksquare & -x\blacksquare \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & 2 \\ 1 & -x & y^2+x \\ 0 & 1 & -x+y-1 \end{pmatrix}$$

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare\blacksquare \\ -x & \blacksquare & \blacksquare\blacksquare \\ \blacksquare\blacksquare & -x\blacksquare & \blacksquare\blacksquare \\ \blacksquare & \blacksquare & -x\blacksquare \end{pmatrix}$$

So finally we obtain a "canonical" Hilbert-Burch matrix:

$$X + A = \begin{pmatrix} y^2 - 1 & 0 & 1 \\ -x + y & y + 1 & 2 \\ 1 & -x + 1 & y^2 + y - 1 \\ 0 & 1 & -x + y - 1 \end{pmatrix} \in \mathscr{A}_{I_0}.$$

PARAMETRIZATIONS

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ -x & \blacksquare & \blacksquare \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \blacksquare & -x \end{pmatrix}$$

So finally we obtain a "canonical" Hilbert-Burch matrix:

$$X + A = \begin{pmatrix} y^2 - 1 & 0 & 1 \\ -x + y & y + 1 & 2 \\ 1 & -x + 1 & y^2 + y - 1 \\ 0 & 1 & -x + y - 1 \end{pmatrix} \in \mathscr{A}_{I_0}.$$