Hilbert Function and Betti Numbers of Algebras with Lefschetz Property of Order m

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Main Goal:

- Characterize the Hilbert function of algebras with the Lefschetz property m times.

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Some results in this direction

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T. Harima, J.C. Migliore, U. Nagel, J. Watanabe

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Generic initial ideals, graded Betti numbers and k-Lefschetz properties,
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- h_{A} - the Hilbert function of A

  $h_d(\mathrm{A}) = \dim_{\mathrm{K}}(\mathrm{A_d})$.

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ℓ_i is a WLE for $\mathrm{A}/(\ell_1, \ldots, \ell_{i-1})$, $\forall\ i \in 1, \ldots, m$.

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h is a **m-times weak Lefschetz O − sequence** if:

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- the sequence $1, h_1 - h_0, \ldots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

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$$h_{\mathrm{A}/(\ell_1)} : 1, h_1 - h_0, \ldots, h_k - h_{k-1}.$$

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$$\beta_{ij}(\mathrm{R}/\mathrm{I}) \leq \beta_{ij}(\mathrm{R}/\mathcal{W}_m(h)) , \qquad \forall i, j \geq 0.$$

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   (b) I is componentwise linear and the ideal $\rho_{n-m}(\mathrm{Gin}(\mathrm{I}))$ is Gotzmann in $\mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{n}-\mathrm{m}}]$.

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   where: $\rho_i : \mathrm{K}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{n}] \longrightarrow \mathrm{K}[\mathrm{x}_1, \dots, \mathrm{x}_\mathrm{i}]$, with:

$$\rho_i(\mathrm{x}_j) = \begin{cases} \mathrm{x}_j & \text{if } j \leq i \\ 0 & \text{if } j > i. \end{cases}$$

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   - $\mathrm{I}_{\leq k_1}$ is the ideal of finite set of rational points in $\mathbb{P}_K^{n-1}$.

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$\Delta^i h$ is a $(m-i)$-times weak Lefschetz O-sequence.

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Technical proof...

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This ensures that we can proceed in the same way, that is by adding in each degree where it is needed the largest in rev-lex order monomials.

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After at most $s - k$ steps we will obtain an ideal $\mathcal{W}_1(h)$ such that:

$$h_{\mathrm{R}/\mathcal{W}_1(h)} = h.$$

Example

Let $h : 1, 4, 7, 8, 6, 3, 1$. Then we have $\Delta h : 1, 3, 3, 1$.

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For $d \geq 4$, the monomials in $\mathrm{R} \setminus \mathrm{I}_1$ are:

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In order to apply induction we also need to show that $\mathcal{W}_1(h)$ is strongly stable.

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Everything works because we only used the fact that Lex(Δh) is strongly stable.

The choice of Lex(Δh) as a starting point is useful for obtaining maximal Betti numbers.

$\mathrm{R}/\mathcal{W}_m(h)$ has the WLP m times

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Lemma

If I *is a strongly stable ideal of* $\mathrm{R} = \mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$*, then T.F.A.E:*

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If I *is a strongly stable ideal of* $\mathrm{R} = \mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$*, then T.F.A.E:*

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   (c) *If* $M \in \mathrm{Gens}(I)$ *is divisible by* x_n, *then* $\deg(M) \geq k+1$.

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Our construction satisfies (a), (b) and (c), so it has the WLP.

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It is also easy to check that x_n is a WLE for $\mathrm{R}/\mathcal{W}_m(h)$.

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It is also easy to check that x_n is a WLE for $\mathrm{R}/\mathcal{W}_m(h)$.

As $\mathrm{R}/\mathcal{W}_m(h) + (\mathrm{x}_n) = \mathrm{R}'/\mathcal{W}_{m-1}(\Delta h)$, which by induction has the WLP $(m-1)$ times, we have that $\mathrm{R}/\mathcal{W}_m(h)$ has m-times the WLP.

$R/\mathcal{W}_m(h)$ has the WLP m times

The proof of the Lemma is based on tahe following result:

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Lemma (A. Wiebe)

If I *is a strongly stable ideal of* $\mathrm{R} = \mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ *then:*

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The proof of the Lemma is based on tahe following result:

Lemma (A. Wiebe)

If I *is a strongly stable ideal of* $R = K[x_1, \ldots, x_n]$ *then:*

$$R/I \text{ has the WLP} \iff x_n \text{ is a WLE for } R/I.$$

$\mathrm{R}/\mathcal{W}_m(h)$ has maximal Betti numbers

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Proposition

If R/I *is an Artinian* $\mathbb{K}$*-algebra with Hilbert function* h *and m-times the WLP then:*

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   Let $\mathrm{I},\ \mathrm{J}$ *be strongly stable ideals with the same Hilbert function. Assume that* $m_{\leq i}(\mathrm{I}_j) \leq m_{\leq i}(\mathrm{J}_j), \ \forall i, j \geq 0$. *Then one has:*

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For a monomial $M = \mathrm{x}_1^{a_1} \ldots \mathrm{x}_n^{a_n}$ in $\mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ we define:

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For a set of monomials $A \subset \mathrm{K}[\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}]$ and for $i = 1, \ldots, n$ we write:

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and from: $m_j(\mathrm{I}) \leq m_j(\mathcal{W}_m(h))$, $\forall\ j > 0$.