Kalai's Conjecture and Beyond

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C-, M. Varbaro
On the h-vectors of Cohen-Macaulay Flag Complexes,
http://arxiv.org/abs/1004.0170 (to appear in Mathematica Scandinavica)

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On the h-vectors of Cohen-Macaulay Flag Complexes,
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D. Cook II, U. Nagel,
Cohen-Macaulay Graphs and Face Vectors of Flag Complexes,
SIAM J. Discrete Math. **26** , no. 1, 89–101 (2012).

Introduction

$$\left\{ \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras} \right\} \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

CM = Cohen-Macaulay

Introduction

$$\left\{ \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras} \right\} \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

characterized by Macaulay

CM = Cohen-Macaulay

Introduction

$$\left\{ \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras} \right\} \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

characterized by Macaulay

characterized by Schützenberger, Kruskal and Katona

CM = Cohen-Macaulay

I n t r o d u c t i o n

$$\left\{ \begin{array}{r} \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras} \\ \text{with extra properties} \end{array} \right\} \left\{ \begin{array}{r} \textit{f-vectors of simplicial complexes} \\ \text{with extra properties} \end{array} \right\}$$

CM = Cohen-Macaulay

Introduction

$$\left\{ \begin{array}{r} \text{quadratic, special system of parameters} \\ \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras} \\ \text{with extra properties} \end{array} \right\} \left\{ \begin{array}{r} \textit{f-vectors of simplicial complexes} \\ \text{with extra properties} \end{array} \right\}$$

CM = Cohen-Macaulay

Introduction

$$\left\{\begin{array}{r}\text{quadratic, special system of parameters}\\ \textit{h-vectors of CM standard graded } \mathbb{K}\textit{-algebras}\\ \text{with extra properties}\end{array}\right\} \left\{\begin{array}{r}\text{flag, balanced}\\ \textit{f-vectors of simplicial complexes}\\ \text{with extra properties}\end{array}\right\}$$

CM = Cohen-Macaulay

I n t r o d u c t i o n

Conjecture 1 (Kalai)

$$\left\{ \textit{h-vectors of CM flag simplicial complexes} \right\} \subset \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

CM = Cohen-Macaulay

Introduction

Conjecture 1 (Kalai)

$$\left\{ \textit{h-vectors of CM flag simplicial complexes} \right\} \subset \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

$$\Uparrow$$

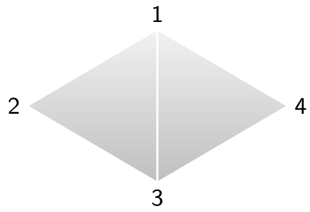
Conjecture (Eisenbud, Green, Harris)

$$\left\{ \textit{h-vectors of quadratic artinian } \mathbb{K}\textit{-algebras} \right\} \subset \left\{ \textit{f-vectors of simplicial complexes} \right\}$$

CM = Cohen-Macaulay

f-vectors and $h-$vectors

Δ :



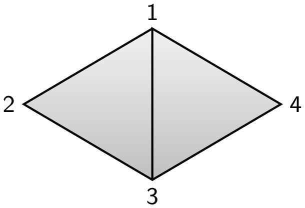
{1,2,3}, {1,3,4}

faces

Simplicial complex = a collection of subsets of a given set, closed under taking subsets.

f-vectors and h-vectors

Δ :



$\{1,2,3\}$, $\{1,3,4\}$

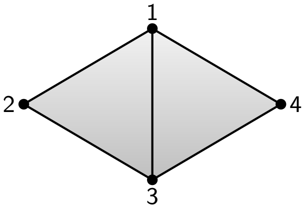
$\{1,2\}$, $\{1,3\}$,$\{1,4\}$,$\{2,3\}$,$\{3,4\}$

faces

Simplicial complex $=$ a collection of subsets of a given set, closed under taking subsets.

f-vectors and $h-$vectors

Δ :



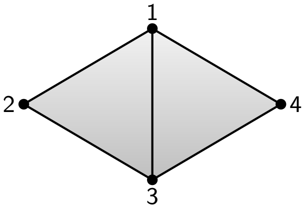
{1,2,3}, {1,3,4}
{1,2}, {1,3},{1,4},{2,3},{3,4}
{1}, {2}, {3},{4}

faces

Simplicial complex = a collection of subsets of a given set, closed under taking subsets.

f-vectors and h-vectors

Δ : $\emptyset$



{1,2,3}, {1,3,4}
{1,2}, {1,3},{1,4},{2,3},{3,4}
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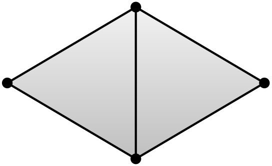
$\emptyset$
faces

Simplicial complex = a collection of subsets of a given set, closed under taking subsets.

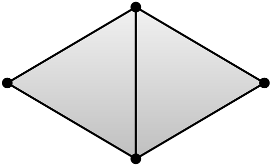
f-vectors and $h-$vectors

Δ : $\emptyset$ 

$f(\Delta) =$

f-vectors and $h-$vectors

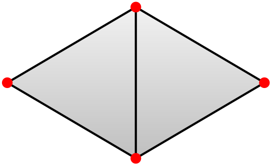
$\Delta :$ $\emptyset$



$f(\Delta) = (1,$

f-vectors and $h-$vectors

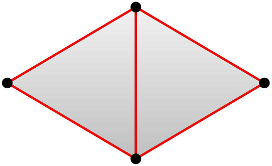
Δ :



$f(\Delta) = (1, 4,$

f-vectors and $h-$vectors

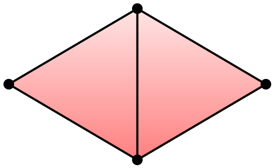
Δ :



$f(\Delta) = (1, 4, 5,$

f-vectors and $h-$vectors

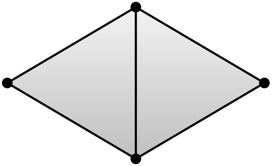
Δ :



$f(\Delta) = (1, 4, 5, 2)$

f-vectors and $h-$vectors

Δ :

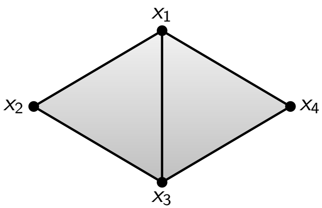


$f(\Delta) = (1, 4, 5, 2)$

$h(\Delta) =$

f-vectors and $h-$vectors

Δ :



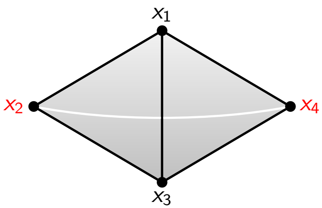
$f(\Delta) = (1, 4, 5, 2)$

$h(\Delta) =$

$R = \mathbb{K}[x_1, x_2, x_3, x_4] \supset I_\Delta$

f-vectors and $h-$vectors

Δ :



$f(\Delta) = (1, 4, 5, 2)$

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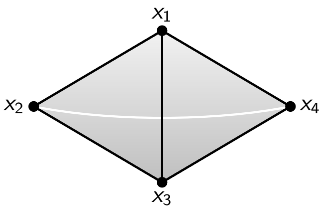
f-vectors and $h-$vectors

Δ :



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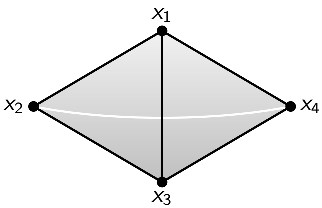
R/I

f-vectors and $h-$vectors

Δ :



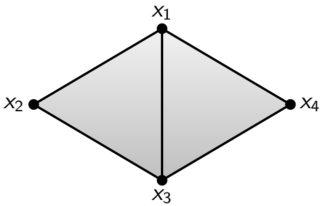
$f(\Delta) = (1, 4, 5, 2)$

$h(\Delta) =$

$R = \mathbb{K}[x_1, x_2, x_3, x_4] \supset I_\Delta = (x_2 x_4)$ $\quad$ $HS_{R/I}(t) = \frac{1 + t}{(1-t)^2}$

f-vectors and h-vectors

Δ :



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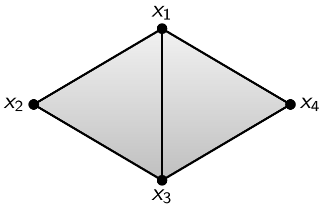
f-vectors and h-vectors

Δ :



$f(\Delta) = (1, 4, 5, 2)$

$h(\Delta) = (1, 1)$

$R = \mathbb{K}[x_1, x_2, x_3, x_4] \supset I_\Delta = (x_2 x_4) \qquad HS_{R/I}(t) = \frac{1 \ +1t}{(1-t)^2}$

f-vectors and $h-$vectors

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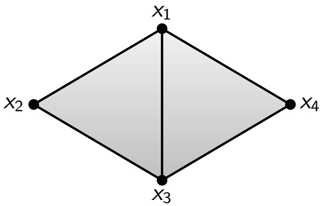


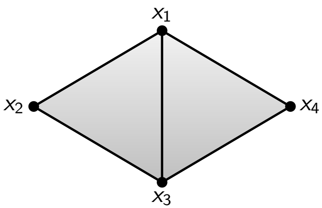
$f(\Delta) = (1, 4, 5, 2)$

$h(\Delta) = (1, 1)$

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-i}{i-j} f_{j-1}$$

f-vectors and $h-$vectors

Δ :



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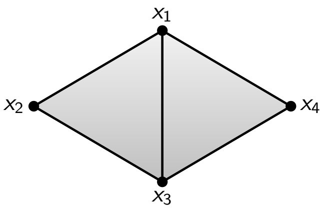

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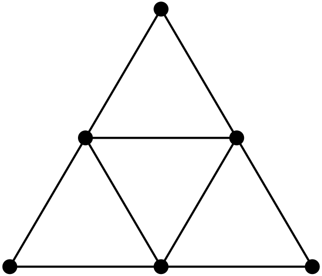
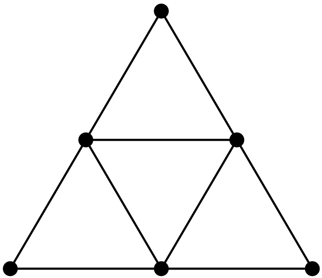
balanced & flag

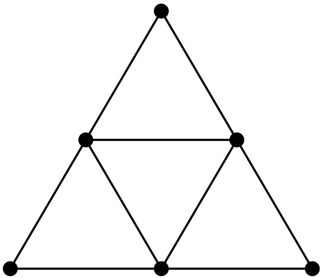


balanced & flag



Δ is balanced if its vertices can be colored in a "minimal" way.

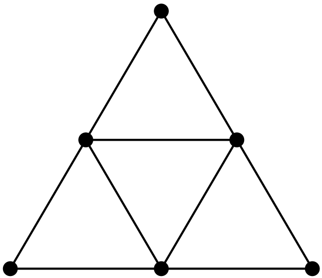
b a l a n c e d & f l a g



Δ is balanced if its vertices can be colored in a "minimal" way.

- no face contains two vertices of the same color

b a l a n c e d & f l a g

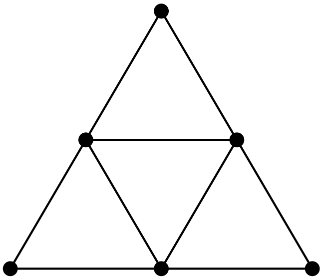


Δ is balanced if its vertices can be colored in a "minimal" way.

- no face contains two vertices of the same color
- using *dimension* $+1$ colors

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

b a l a n c e d & f l a g



dimension $+1 = 2$

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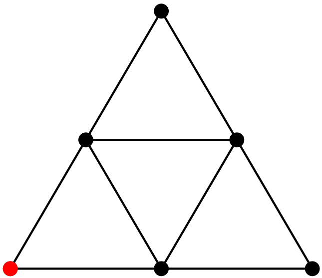
b a l a n c e d & f l a g



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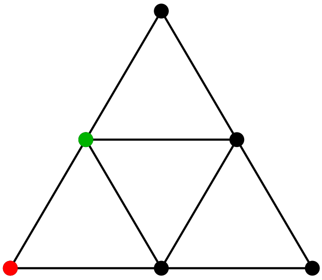
balanced & flag



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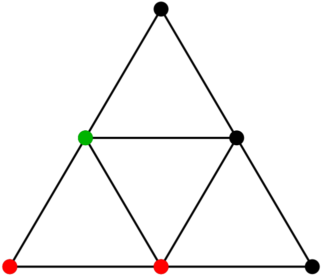
b a l a n c e d & f l a g



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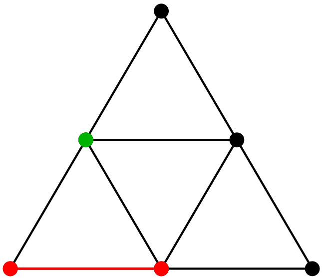
b a l a n c e d & f l a g



dimension $+1 = 2$

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- no face contains two vertices of the same color
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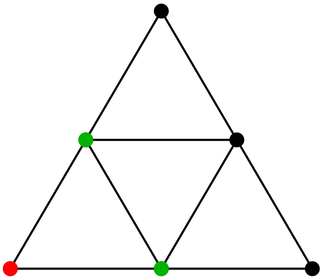
balanced & flag



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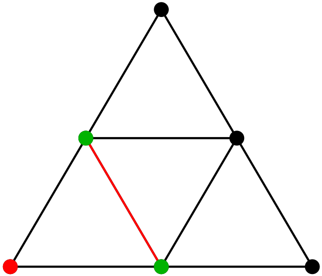

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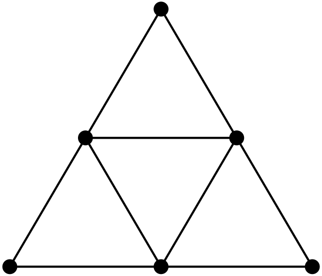


dimension $+1 = 2$

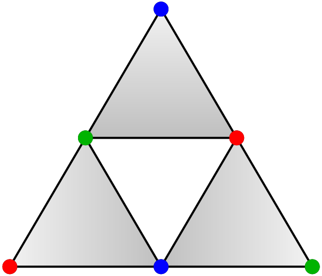
Δ is not balanced

Δ is balanced if its vertices can be colored in a "minimal" way.

- no face contains two vertices of the same color
- using *dimension* $+1$ colors

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

b a l a n c e d & f l a g



dimension $+1 = 3$

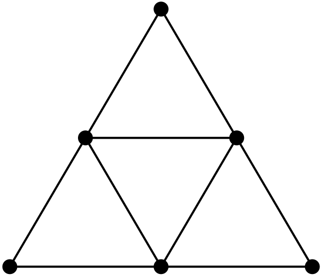
Δ is balanced

Δ is balanced if its vertices can be colored in a "minimal" way.

- no face contains two vertices of the same color
- using *dimension* $+1$ colors

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

balanced & flag

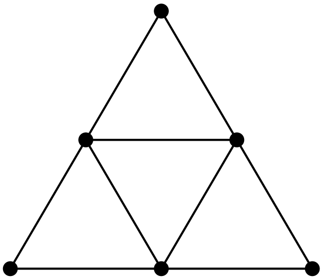


Δ is balanced if its vertices can be colored in a "minimal" way.

Δ is flag if all minimal non-faces have cardinality two.

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

b a l a n c e d & f l a g



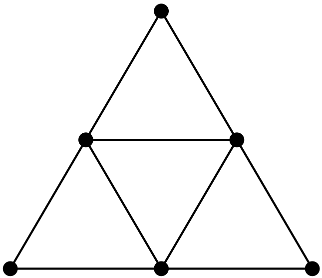
Δ is balanced if its vertices can be colored in a "minimal" way.

Δ is flag if all minimal non-faces have cardinality two.

- all triangles, tetrahedra, etc. are "filled"

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

balanced & flag



Δ is not flag

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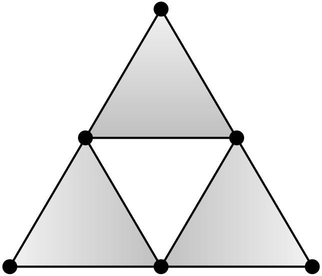
b a l a n c e d & f l a g



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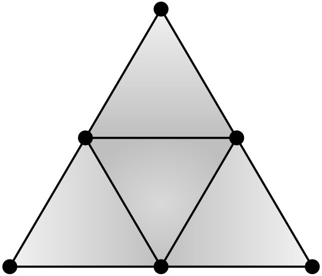

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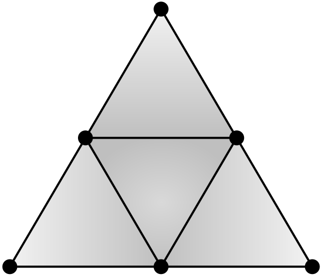


Δ is balanced if its vertices can be colored in a "minimal" way.

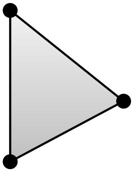
Δ is flag if all minimal non-faces have cardinality two.

Remark *If Δ is flag then I_Δ is generated by quadrics.*

dimension of $\Delta = \max\{|F| - 1 \ : \ F \in \Delta\}$

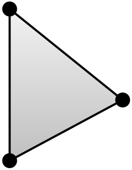
Δ is vertex decomposable if

v e r t e x   d e c o m p o s a b l e



Δ is vertex decomposable if

Δ is a simplex

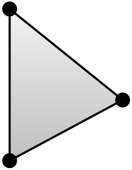


Δ is vertex decomposable if

Δ is a simplex

or

vertex decomposable



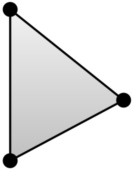
Δ is vertex decomposable if

Δ is a simplex

or

there exists a vertex $v \in \Delta$ such that

(a) $\Delta \setminus v$ and $\text{link}_{\Delta} v$ are vertex decomposable



Δ is vertex decomposable if

Δ is a simplex

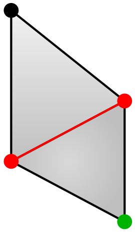
or

there exists a vertex $v \in \Delta$ such that

(a) $\Delta \setminus v$ and $\text{link}_\Delta v$ are vertex decomposable

(b) no facet of $\text{link}_\Delta v$ is a facet of $\Delta \setminus v$

facet = a face which is maximal with respect to inclusion.



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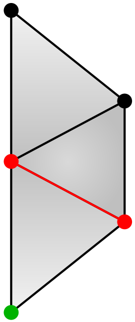
v e r t e x   d e c o m p o s a b l e



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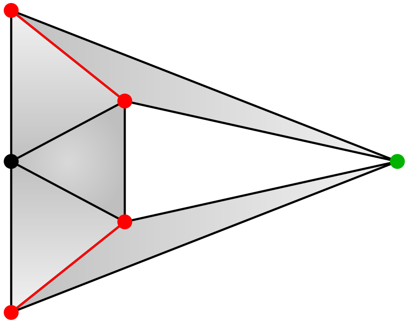
vertex decomposable



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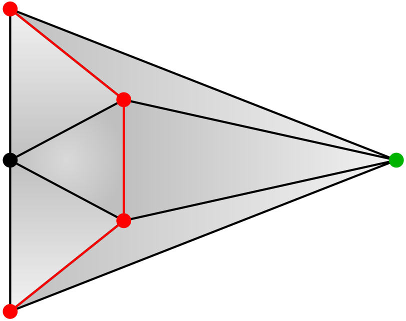

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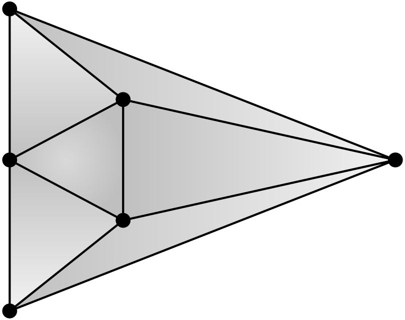

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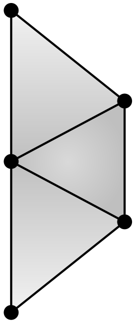

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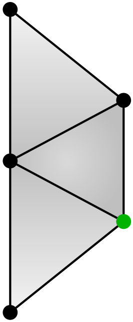

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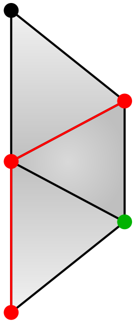

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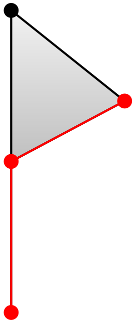

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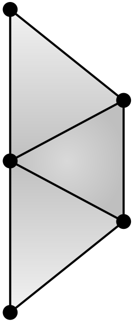

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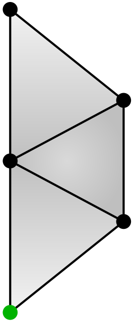

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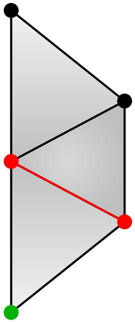

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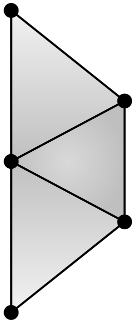

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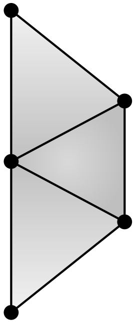
Δ is vertex decomposable $\Rightarrow R/I_\Delta$ is CM.

facet = a face which is maximal with respect to inclusion.



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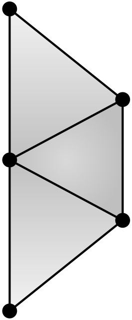
Δ is vertex decomposable $\Rightarrow R/I_\Delta$ is CM. (in this case we say that Δ is CM)

facet = a face which is maximal with respect to inclusion.

vertex decomposable



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Δ is vertex decomposable $\Rightarrow$ Δ is shellable $\Rightarrow$ Δ is CM

facet = a face which is maximal with respect to inclusion.

First result

Conjecture 1 (Kalai)

$\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

First result

Theorem 1 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable flag s.c.*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

First result

Theorem 1 (C-, Varbaro)

{*h-vectors of vertex decomposable flag s.c.*} $\subset$ {*f-vectors of simplicial complexes*}.

Idea of Proof:

Kalai's conjecture: {*h-vectors of CM flag simplicial complexes*} $\subset$ {*f-vectors of simplicial complexes*}.

First result

Theorem 1 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable flag s.c.*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Idea of Proof:

- Use induction: $h(\Delta \setminus v) = f(\Gamma_1)$ and $h(\mathrm{link}_\Delta v) = f(\Gamma_2)$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

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First result

Theorem 1 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable flag s.c.*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

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- Construct $\Gamma = \Gamma_1 *_{\Gamma_2} u$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

First result

Theorem 1 (C-, Varbaro)

$\{h$-vectors of vertex decomposable flag s.c.$\} \subset \{f$-vectors of simplicial complexes$\}$.

Idea of Proof:

- Use induction: $h(\Delta \setminus v) = f(\Gamma_1)$ and $h(\mathrm{link}_\Delta v) = f(\Gamma_2)$.
- Prove that we may take $\Gamma_2 \subset \Gamma_1$ (hardest part).
- Construct $\Gamma = \Gamma_1 *_{\Gamma_2} u$.
- Using recursive formulas for the h-vectors and f-vectors conclude that

$$h(\Delta) = f(\Gamma).$$

Kalai's conjecture: $\{h$-vectors of CM flag simplicial complexes$\} \subset \{f$-vectors of simplicial complexes$\}$.

f-vectors of flag simplicial complexes

Remark

$\{$*f-vectors of flag s.c.*$\} = \{$*h-vectors of quadratic monomial artinian algebras*$\}$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Remark

$\{f$-vectors of flag s.c.$\} = \{h$-vectors of quadratic monomial artinian algebras$\}$.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \ldots, x_n^2)).$$

Kalai's conjecture: $\{h$-vectors of CM flag simplicial complexes$\} \subset \{f$-vectors of simplicial complexes$\}$.

Remark

$\{f\text{-vectors of flag s.c.}\} = \{h\text{-vectors of quadratic monomial artinian algebras}\}$.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \ldots, x_n^2)).$$

Proposition

$\{f\text{-vectors of flag s.c.}\} \subseteq \{h\text{-vectors of vertex decomposable balanced flag s.c.}\}$.

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

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$$f(\Gamma) = h(S/I_\Gamma + (x_1 y_1 \ldots, x_n y_n))$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}.$

f-vectors of flag simplicial complexes

Remark

{f-vectors of flag s.c.} = {h-vectors of quadratic monomial artinian algebras}.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \ldots, x_n^2)).$$

Proposition

{f-vectors of flag s.c.} ⊆ {h-vectors of vertex decomposable balanced flag s.c.}.

$$f(\Gamma) = h(S/I_\Delta))$$

Kalai's conjecture: {h-vectors of CM flag simplicial complexes} ⊂ {f-vectors of simplicial complexes}.

Remark

$\{$*f-vectors of flag s.c.*$\} = \{$*h-vectors of quadratic monomial artinian algebras*$\}$.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \ldots, x_n^2)).$$

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$\{$*f-vectors of flag s.c.*$\} \subseteq \{$*h-vectors of vertex decomposable balanced flag s.c.*$\}$.

$$f(\Gamma) = h(S/I_\Delta))$$

with $\Delta =$ vertex decomposable,

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Remark

$\{f$*-vectors of flag s.c.*$\} = \{h$*-vectors of quadratic monomial artinian algebras*$\}$.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \ldots, x_n^2)).$$

Proposition

$\{f$*-vectors of flag s.c.*$\} \subseteq \{h$*-vectors of vertex decomposable balanced flag s.c.*$\}$.

$$f(\Gamma) = h(S/I_\Delta))$$

with Δ = vertex decomposable, flag

Kalai's conjecture: $\{h$*-vectors of CM flag simplicial complexes*$\} \subset \{f$*-vectors of simplicial complexes*$\}$.

Remark

{f-vectors of flag s.c.} = {h-vectors of quadratic monomial artinian algebras}.

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \dots, x_n^2)).$$

Proposition

{f-vectors of flag s.c.} ⊆ {h-vectors of vertex decomposable balanced flag s.c.}.

$$f(\Gamma) = h(S/I_\Delta))$$

with Δ = vertex decomposable, flag and balanced.

Kalai's conjecture: {h-vectors of CM flag simplicial complexes} ⊂ {f-vectors of simplicial complexes}.

Remark

$\{f\text{-vectors of flag s.c.}\} = \{h\text{-vectors of quadratic monomial artinian algebras}\}.$

$$f(\Gamma) = h(R/I_\Gamma + (x_1^2, \dots, x_n^2)).$$

Proposition

$\{f\text{-vectors of flag s.c.}\} \subseteq \{h\text{-vectors of vertex decomposable balanced flag s.c.}\}.$

$$f(\Gamma) = h(S/I_\Delta))$$

with Δ = vertex decomposable, flag and balanced.

Conjecture 1 (C-, Varbaro)

$\{f\text{-vectors of flag s.c.}\} = \{h\text{-vectors of vertex decomposable balanced flag s.c.}\}.$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}.$

f-vectors of flag simplicial complexes

Definition

Δ is quasi-flag if

Kalai's conjecture: {*h-vectors of CM flag simplicial complexes*} $\subset$ {*f-vectors of simplicial complexes*}.

Our 1st conjecture: {*h-vectors of vertex decomposable balanced flag s.c.*} $=$ {*f-vectors of flag s.c.*}.

Definition

Δ is quasi-flag if $\Delta = \{\emptyset\}$ or

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

Definition

Δ is quasi-flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

Kalai's conjecture: $\{$h-vectors of CM flag simplicial complexes$\} \subset \{$f-vectors of simplicial complexes$\}$.

Our 1st conjecture: $\{$h-vectors of vertex decomposable balanced flag s.c.$\} = \{$f-vectors of flag s.c.$\}$.

f-vectors of flag simplicial complexes

Definition

Δ is quasi-flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

(b) $\mathrm{link}_\Delta v$ is an *induced* subcomplex of Δ and has the f-vector of a quasi-flag simplicial complex.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

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(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

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Remark

1. *flag* $\Rightarrow$ *quasi-flag*

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

Definition

Δ is flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ is a flag simplicial complex.

(b) $\mathrm{link}_\Delta v$ is a flag *induced* subcomplex of Δ.

Remark

1. *flag $\Rightarrow$ quasi-flag*

Kalai's conjecture: $\{$h-vectors of CM flag simplicial complexes$\} \subset \{$f-vectors of simplicial complexes$\}$.

Our 1st conjecture: $\{$h-vectors of vertex decomposable balanced flag s.c.$\} = \{$f-vectors of flag s.c.$\}$.

Definition

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(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

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Remark

1. *flag* $\Rightarrow$ *quasi-flag*
2. $\{$*f-vectors of s.c.*$\} \not\subseteq \{$*f-vectors of quasi-flag s.c.*$\}$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

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Δ is quasi-flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

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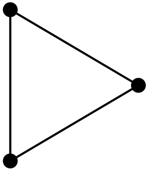
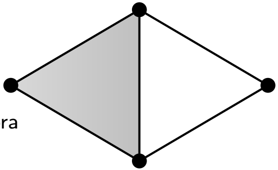
Remark

1. *flag* $\Rightarrow$ *quasi-flag*
2. $\{$*f-vectors of s.c.*$\} \not\subseteq \{$*f-vectors of quasi-flag s.c.*$\}$.



$(1, 3, 3)$

$(1, 4, 5, 1)$



Neither the f-vector of a quasi-flag complex

Nor the h-vector of a quadratic artinian algebra

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

f-vectors of flag simplicial complexes

Definition

Δ is quasi-flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

(b) $\text{link}_\Delta v$ is an *induced* subcomplex of Δ and has the f-vector of a quasi-flag simplicial complex.

Remark

1. *flag* $\Rightarrow$ *quasi-flag*
2. $\{f\text{-vectors of s.c.}\} \not\subseteq \{f\text{-vectors of quasi-flag s.c.}\}$.

We are not aware of any example of quasi-flag simplicial complex whose f-vector is not that of a flag complex.

$$\{f\text{-vectors of quasi-flag complexes}\} \supseteq \{f\text{-vectors of flag complexes}\}.$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

Definition

Δ is quasi-flag if $\Delta = \{\emptyset\}$ or if there exists $v \in \Delta$ such that

(a) $\Delta \setminus v$ has the f-vector of a quasi-flag simplicial complex.

(b) $\text{link}_\Delta v$ is an *induced* subcomplex of Δ and has the f-vector of a quasi-flag simplicial complex.

Remark

1. *flag* $\Rightarrow$ *quasi-flag*
2. $\{f$*-vectors of s.c.*$\} \not\subseteq \{f$*-vectors of quasi-flag s.c.*$\}$.

We are not aware of any example of quasi-flag simplicial complex whose f-vector is not that of a flag complex.

$$\{f\text{-vectors of quasi-flag complexes}\} \stackrel{?}{=} \{f\text{-vectors of flag complexes}\}.$$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{f$*-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{f$*-vectors of flag s.c.*$\}$.

Theorem 2 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} \subseteq \{$*f-vectors of quasi-flag s.c.*$\}$.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

f-vectors of flag simplicial complexes

Theorem 2 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} \subseteq \{$*f-vectors of quasi-flag s.c.*$\}$.

Theorem 3 (C-, Varbaro)

$$\left\{\begin{array}{c}\textit{h-vectors of } (d-1)\textit{-dimensional}\\ \textit{vertex decomposable balanced flag}\\ \textit{simplicial complexes on } 2d \textit{ vertices}\end{array}\right\} = \left\{\begin{array}{c}\textit{f-vectors of flag}\\ \textit{simplicial complexes}\\ \textit{on } d \textit{ vertices}\end{array}\right\}.$$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

f-vectors of flag simplicial complexes

Theorem 2 (C-, Varbaro)

$\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} \subseteq \{$*f-vectors of quasi-flag s.c.*$\}$.

Theorem 3 (C-, Varbaro)

$$\left\{ \begin{array}{c} \textit{h-vectors of } (d-1)\textit{-dimensional} \\ \textit{vertex decomposable balanced flag} \\ \textit{simplicial complexes on } 2d \textit{ vertices} \end{array} \right\} = \left\{ \begin{array}{c} \textit{f-vectors of flag} \\ \textit{simplicial complexes} \\ \textit{on } d \textit{ vertices} \end{array} \right\}.$$

If Δ is flag, then $I_\Delta = I(G_\Delta)$, the edge ideal of a graph G_Δ.

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

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f-vectors of flag simplicial complexes

Theorem 2 (C-, Varbaro)

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Theorem 3 (C-, Varbaro)

$$\left\{ \begin{array}{c} h\text{-vectors of } (d-1)\text{-dimensional} \\ \text{vertex decomposable balanced flag} \\ \text{simplicial complexes on } 2d \text{ vertices} \end{array} \right\} = \left\{ \begin{array}{c} f\text{-vectors of flag} \\ \text{simplicial complexes} \\ \text{on } d \text{ vertices} \end{array} \right\}.$$

If Δ is flag, then $I_\Delta = I(G_\Delta)$, the edge ideal of a graph G_Δ.

$$\Delta (d-1)\text{-dimensional on } 2d \text{ vertices} \Leftarrow G_\Delta \text{ is bipartite.}$$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{\textit{h-vectors of CM flag s.c.}\} = \{\textit{f-vectors of flag s.c.}\}.$$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{\textit{h-vectors of CM flag s.c.}\} = \{\textit{f-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{\textit{f-vectors of CM flag s.c.}\} \subseteq \{\textit{f-vectors of CM balanced s.c.}\}.$$

$h(\text{CM flag}) = f(\text{flag})$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

h(CM flag) $= f$(flag)
$= f$(balanced) by Frohmader

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

h(CM flag) $= f$(flag)
$= f$(balanced) by Frohmader
$= h$(CM balanced, of smaller dimension) by Björner, Frankl and Stanley

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

h(CM flag) $=$ f(flag)
$=$ f(balanced) by Frohmader
$=$ h(CM balanced, of smaller dimension) by Björner, Frankl and Stanley
$=$ h(CM balanced of the same dimension) adding cone points

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

h(CM flag) $= f$(flag)
$= f$(balanced) by Frohmader
$= h$(CM balanced, of smaller dimension) by Björner, Frankl and Stanley
$= h$(CM balanced of the same dimension) adding cone points

$$\Rightarrow f(\text{CM flag}) = f(\text{CM balanced}).$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

$$\Downarrow$$

Conjecture

$$\{h\text{-vectors of CM flag s.c.}\} \subseteq \{h\text{-vectors of CM balanced s.c.}\}.$$

Because

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-i}{i-j} f_{j-1}.$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{\textit{h-vectors of CM flag s.c.}\} = \{\textit{f-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{\textit{f-vectors of CM flag s.c.}\} \subseteq \{\textit{f-vectors of CM balanced s.c.}\}.$$

$$\Downarrow$$

Conjecture

$$\{\textit{h-vectors of CM flag s.c.}\} \subseteq \{\textit{h-vectors of CM balanced s.c.}\}.$$

By a theorem of Björner, Frankl and Stanley we have:

$$\{h\text{-vectors of CM balanced s.c}\} \subseteq \{f\text{-vectors of s.c.}\}.$$

Kalai's conjecture: $\{$*h-vectors of CM flag simplicial complexes*$\} \subset \{$*f-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{$*h-vectors of vertex decomposable balanced flag s.c.*$\} = \{$*f-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

$$\Downarrow$$

Conjecture

$$\{h\text{-vectors of CM flag s.c.}\} \subseteq \{h\text{-vectors of CM balanced s.c.}\}.$$

By a theorem of Björner, Frankl and Stanley we have:

$$\{h\text{-vectors of CM balanced s.c}\} \subseteq \{f\text{-vectors of s.c.}\}.$$

Kalai's conjecture: $\{h$*-vectors of CM flag simplicial complexes*$\} \subset \{f$*-vectors of simplicial complexes*$\}$.

Our 1st conjecture: $\{h$*-vectors of vertex decomposable balanced flag s.c.*$\} = \{f$*-vectors of flag s.c.*$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Conjecture 2 (Kalai)

$$\{f\text{-vectors of CM flag s.c.}\} \subseteq \{f\text{-vectors of CM balanced s.c.}\}.$$

$$\Downarrow$$

Conjecture 1 (Kalai)

$$\{h\text{-vectors of CM flag s.c.}\} \subset \{f\text{-vectors of s.c.}\}.$$

Kalai's conjecture: $\{h$-vectors of CM flag simplicial complexes$\} \subset \{f$-vectors of simplicial complexes$\}$.

Our 1st conjecture: $\{h$-vectors of vertex decomposable balanced flag s.c.$\} = \{f$-vectors of flag s.c.$\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Theorem 4 (C-, Varbaro)

$$\left\{ \begin{array}{c} h\text{-vectors of } (d-1)\text{-dimensional} \\ \text{CM flag s.c. on } 2d \text{ vertices} \\ \text{without cone points} \end{array} \right\} = \left\{ \begin{array}{c} f\text{-vectors of flag} \\ \text{simplicial complexes} \\ \text{on } d \text{ vertices} \end{array} \right\}.$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{\textit{h-vectors of CM flag s.c.}\} = \{\textit{f-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Theorem 4 (C-, Varbaro)

$$\left\{ \begin{array}{c} \textit{h-vectors of } (d-1)\textit{-dimensional} \\ \textit{CM flag s.c. on } 2d \textit{ vertices} \\ \textit{without cone points} \end{array} \right\} = \left\{ \begin{array}{c} \textit{f-vectors of flag} \\ \textit{simplicial complexes} \\ \textit{on } d \textit{ vertices} \end{array} \right\}.$$

This is the “first” case of our Conjecture 2 in the following sense:

Kalai's conjecture: $\{\textit{h-vectors of CM flag simplicial complexes}\} \subset \{\textit{f-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{\textit{h-vectors of vertex decomposable balanced flag s.c.}\} = \{\textit{f-vectors of flag s.c.}\}$.

A natural generalization

Conjecture 2 (C-, Varbaro)

$$\{h\text{-vectors of CM flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}.$$

$$\Downarrow$$

Theorem 4 (C-, Varbaro)

$$\left\{ \begin{array}{c} h\text{-vectors of } (d-1)\text{-dimensional} \\ \text{CM flag s.c. on } 2d \text{ vertices} \\ \text{without cone points} \end{array} \right\} = \left\{ \begin{array}{c} f\text{-vectors of flag} \\ \text{simplicial complexes} \\ \text{on } d \text{ vertices} \end{array} \right\}.$$

This is the "first" case of our Conjecture 2 in the following sense:

Remark

If Δ is $(d-1)$-dimensional, flag, on n vertices and without cone points, then

$$n \geq 2d.$$

Kalai's conjecture: $\{h\text{-vectors of CM flag simplicial complexes}\} \subset \{f\text{-vectors of simplicial complexes}\}$.

Our 1st conjecture: $\{h\text{-vectors of vertex decomposable balanced flag s.c.}\} = \{f\text{-vectors of flag s.c.}\}$.

Thank you!