Parametrizations of Ideals in $K[x, y]$

Alexandru Constantinescu
Mathematisches Institut, Universität Basel

WHAT IS GIVEN

- A field K, an **artinian monomial** ideal $I_0 \subset R = K[x, y]$ and $J_0 = I_0 S \subset S = K[x, y, z]$,
- The **degree reversed-lexicographic term order** (DRL) induced by $x > y > z$.

THE OBJECTS OF STUDY

- The **Hilbert scheme** of n points in the projective plane: $Hilb^n(\mathbb{P}^2)$
- The **Hilbert strata** $\mathbb{G}(H) = \{I \subset S \ : \ I \text{ homogeneous with } H_{S/I} = H\} \subseteq Hilb^n(\mathbb{P}^2)$
- The **Betti strata**: $\forall\, 0 \leq j, u \in \mathbb{Z}$ and a vector $\beta = (\beta_1, \ldots, \beta_j, \ldots)$ with integral entries we define:

$$V(H, j, u) = \{J \in \mathbb{G}(H) \ : \ \beta_{0,j}(J) = u\}$$

$$V(H, j, \geq u) = \{J \in \mathbb{G}(H) \ : \ \beta_{0,j}(J) \geq u\}$$

$$V(H, \beta) = \cap_j V(H, j, \beta_j)$$

$$V(H, \geq \beta) = \cap_j V(H, j, \geq \beta_j)$$

- The **Gröbner cells** $V_{hom}(J_0) = \{I \subset S \ : \ I \text{ homogenous with } \mathrm{in}(I) = J_0\} \subseteq \mathbb{G}(H_{S/J_0})$
- The **affine Gröbner cells** $V(I_0) = \{I \subset R \ : \ \mathrm{in}(I) = I_0\} \subseteq Hilb^n(\mathbb{A}^2)$.

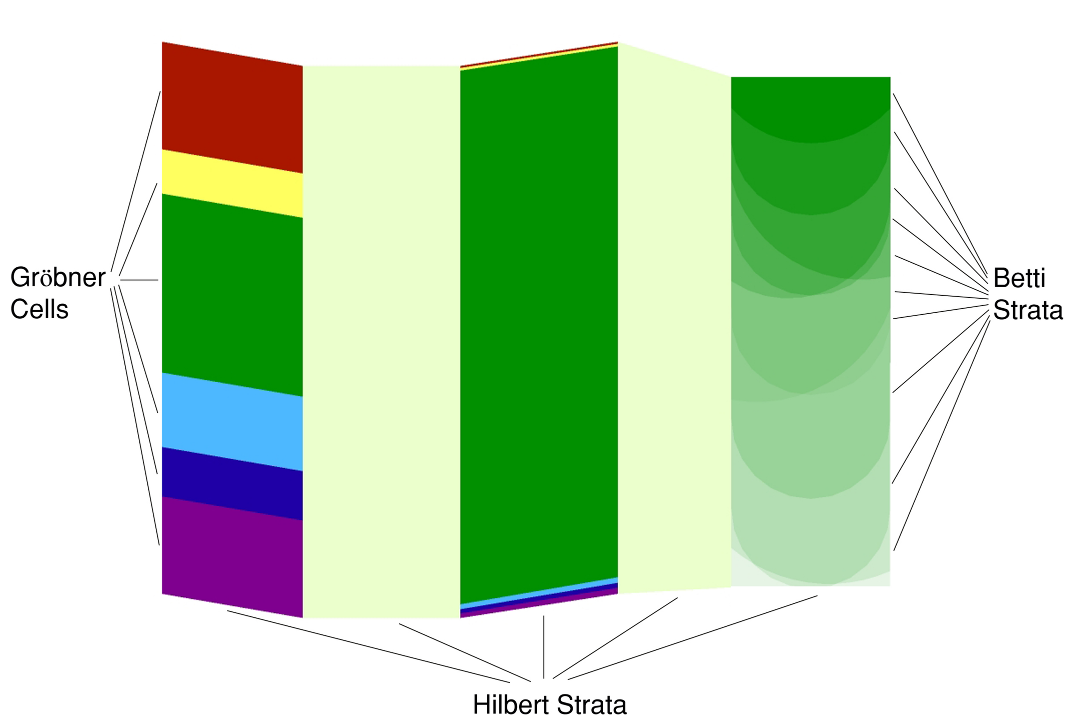
PROBLEMS

Both $V(I_0)$ and $V_{hom}(J_0)$ have natural structures of affine varieties. General results imply that for ideals of $\mathbf{K[x,y]}$ both $\mathbf{V(I_0)}$ and $\mathbf{V_{hom}(I_0)}$ are **affine spaces**. However no parametrization of this spaces was known. In general, for ideals in three or more variables we do not obtain affine spaces.

Problem 1: **find a parametrization** of $V(I_0)$.

Problem 2: **extend it** to $V_{hom}(J_0)$ by homogenization.

Problem 3: **use this parametrization** to study the Betti strata of $\mathbb{G}(H_{S/J_0})$.



$Hilb^n(\mathbb{P}^2)$

THE TECHNICAL DETAILS

Choose a **particular set of generators** of I_0:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}),$$

Notice that $0 = m_0 \leq m_1 \leq \ldots \leq m_t$ and define

$$d_i = m_i - m_{i-1}$$

$$u_{i,j} = i - j + m_j - m_{i-1}$$

Define the set of matrices

$$\mathcal{A}_{I_0} = \{(a_{i,j}) \in \mathcal{M}_{t,t+1}(K[y]) \ : \ \text{satisfying (1)}\ \}$$

where:

$$\deg(a_{i,j}) \leq \begin{cases} \mathrm{Min}\{u_{i,j} - 1,\ d_i - 1\} & \text{if } i \leq j, \\ \mathrm{Min}\{u_{i,j}\ \ \ \ ,\ d_j - 1\} & \text{if } i > j, \end{cases} \qquad (1)$$

Notice that $\mathcal{A}_{I_0}$ is an affine space.

Choose the following Hilbert-Burch matrix of I_0:

$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

Let $\psi : \mathcal{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$.

RESULTS

The Parametrization

Theorem 1: Let $I_0 \subset R$ be an monomial lex-segment ideal, with $\dim_K(R/I_0) < \infty$. The map of affine varieties $\psi : \mathcal{A}_{I_0} \longrightarrow V(I_0)$ is bijective.

Theorem 2: In the above setting the affine space $\mathcal{A}_{I_0}$ also parametrizes $V_{hom}(J_0)$.

The Hilbert Strata

- In characteristic zero or "large enough" we parametrize a dense subset of the Hilbert stratum:

$$V_{hom}(J_0) \subseteq_{open} \mathbb{G}(H_{S/J_0})$$

- If $(h_0, \ldots, h_s) = H_{R/I_0}$ and $H = H_{S/J_0}$, we obtain the following the dimension formula for $\mathbb{G}(H_{S/J_0})$

$$\dim(\mathbb{G}(H)) = \dim(\mathcal{A}_{I_0}) = \dim_K(R/I_0) + 1 + \sum_{i \geq 1} h_i(h_{i-1} - h_{i-2}),$$

* Different formulas were determined by G.Gotzmann [4], by G. Ellingsrud and S. A. Strømme [2, 3], by A. Iarrobino and V. Kanev [6] and by K. De Naeghel and M. Van den Bergh [7].

The Betti strata

Let $J \in \mathbb{G}(H)$ and set $\beta = (\beta_{0,j}(J))$.

- Each $V(H, j, \geq \beta_j)$ is a determinantal variety.
- The variety $V(H, j, \geq \beta_j)$ coincides with the closure of $V(H, j, \beta_j)$, provided $V(H, j, \beta_j)$ is not empty.
- The variety $V(H, \geq \beta)$ is irreducible and it is the transversal intersection of the $V(H, j, \geq \beta_j)$'s.

Theorem 3: The variety $V(H, \geq \beta)$ is irreducible, it is the closure of $V(H, \beta)$ and has the following codimension in $\mathbb{G}(H)$

$$\sum_j \beta_{1,j}(J)\beta_{0,j}(J).$$

*The codimension formula in Theorem 3 was conjectured by A. Iarrobino in [5] with an indication of a proof. The techniques used here are different.

REMARKS

- The natural variety structure of the Gröbner cells is given by an embedding in a higher dimensional affine space. The equations are obtained from Buchberger's criterium.
- Degree-compatibility is crucial in the homogenization process. As we start with ideals in 2 variables, the extension of the parametrization from $V(I_0)$ to $V_{hom}(J_0)$ is *only possible for* the DRL term order.

POSSIBLE GENERALIZATION

1. Generalize the parametrization to any term order.
2. Compute intersections of Gröbner cells with I_0 fixed and varying the term order.
3. Study the analogous problem in the local case.
4. Determinantal conditions on blocks of constants in $X + A$ determine bounds on the Betti numbers. Study the role played by other "groups" of parameters.

REFERENCES

[1] A. Constnatninescu, *Parametrizations of ideals in $K[x, y]$*, submitted for publication.

[2] G.Ellingsrud, S.A.Strømme, *On the Homology of the Hilbert Scheme of Points in the Plane*, Invent. Math. **87**, 343–352, (1987).

[3] G.Ellingsrud, S.A.Strømme, *On a Cell Decomposition of the Hilbert Scheme of Points in the Plane*, Invent. Math. **91**, 365–370, (1988).

[4] G. Gotzmann *A Stratification of the Hilbert Scheme of Points in the Projective Plane*, Math. Z. **199**, no.4, 539–547, (1988).

[5] A. Iarrobino, *Betti Strata of Height Two Ideals.* Journal of Algebra 285. 835-855. (2005)

[6] A. Iarrobino, V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lect.Not. in Math., **1721**, Springer-Verlag, Berlin, (1999).

[7] K. De Naeghel, M. Van den Bergh, *Ideal classes of three dimensional Artin-Schelter regular algebras*, J. Algebra **283** , no. 1, 399–429, (2005).