Parametrizations of Ideals in $K[x, y]$ and $K[x, y, z]$

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- τ is the degree reverse-lexicographic $(x > y)$ term order,
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- I_0 is lex-segment ideal.

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We prove A. Iarrobino's codimension formula for the Betti strata of codimension two punctual schemes in $\mathbb{P}^2$.

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$$X = \begin{pmatrix} y^{d_1} & 0 & \ldots & 0 \\ -x & y^{d_2} & \ldots & 0 \\ 0 & -x & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & y^{d_t} \\ 0 & 0 & \ldots & -x \end{pmatrix}$$

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Notice that the matrix X is a Hilbert-Burch matrix for I_0.

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Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$.

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Theorem

Let $I_0 \subset R = K[x, y]$ be a monomial lex-segment ideal with $\dim_K(R/I_0) < \infty$.
Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.

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Let $I_0 \subset R$ be a monomial lex-segment ideal. Using the above notation we have the following formula:

$$\dim(V(I_0)) = \dim_K(R/I_0) + 1 + \sum_{i \geq 1} h_i(h_{i-1} - h_{i-2}).$$

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If $J_0 = I_0 K[x, y, z]$, with $I_0 \in K[x, y]$ a lex-segment ideal, τ is the Deg Rev-Lex order induced by $x > y > z$, then $V_h(J_0)$ is again the affine space $\mathscr{A}_{I_0}$.

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Example: $I_0 = (x^2 y^0, xy^3, y^5)$, $u_{i,j} = i - j + m_j - m_{i-1}$,

$$(u_{i,j}) = \begin{pmatrix} 3 & 4 \\ 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} y^2 & +y & y^2 & -1 \\ & y & y & +1 \\ & 1 & & 1 \end{pmatrix}.$$

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\begin{aligned}
f_0 &= x^3 - 2x^2y + y^3 - xy + y^2 - x - 3,\\
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E x a m p l e

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The degrees of the matrix in $\mathscr{A}_{I_0}$ are bounded by:

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To obtain a matrix in $\mathscr{A}_{I_0}$ we operate on the Hilbert-Burch matrix $X + A'$. We will use some reduction moves, which are a sequence of two elementary operations on $X + A'$.

Example

$$\begin{pmatrix} \blacksquare & \blacksquare & \blacksquare \\ -x & \blacksquare & \blacksquare \\ \blacksquare & -x & \blacksquare \\ \blacksquare & \blacksquare & -x \end{pmatrix}$$

$$\begin{pmatrix} y^2-1 & 0 & y^2 \\ -x+y & y & y+3 \\ 1 & -x & y^2+1 \\ 0 & 1 & -x+y \end{pmatrix}$$

Example

$$\begin{pmatrix} \square & \blacksquare & {\color{red}\blacksquare} \\ -x & \blacksquare & {\color{red}\blacksquare} \\ \square & -x & \blacksquare \\ \square & \blacksquare & -x \end{pmatrix}$$

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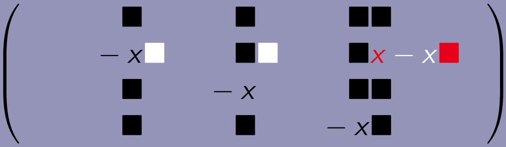

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$$\begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & 2 \\ 1 & -x & y^2+x \\ 0 & 1 & -x+y-1 \end{pmatrix}$$

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So finally we obtain a "canonical" Hilbert-Burch matrix:

$$X + A = \begin{pmatrix} y^2-1 & 0 & 1 \\ -x+y & y+1 & 2 \\ 1 & -x+1 & y^2+y-1 \\ 0 & 1 & -x+y-1 \end{pmatrix} \in \mathscr{A}_{I_0}.$$

Example

$$
\begin{pmatrix}
\blacksquare & \blacksquare & \blacksquare \\
-\,x & \blacksquare & \blacksquare \\
\blacksquare & -\,x & \blacksquare \\
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1 & -x + 1 & y^2 + y - 1 \\
0 & 1 & -x + y - 1
\end{pmatrix} \in \mathscr{A}_{I_0}.
$$