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Combinatorial Structures, Lefschetz Property and Parametrizations of Algebras

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Introduction

In this thesis we study several algebraic problems related to algebras and modules with straightening laws, the Lefschetz property and parametrizations of ideals in polynomial rings with two and three variables. Throughout this thesis we will also present interactions of algebraic properties with geometrical and combinatorial ones.

As the three chapters of this thesis are independent, we will start each of them with some definitions and known results that we will use further on.

In the first chapter we study a problem related to algebras with straightening laws (ASL for short), namely: is the Veronese algebra of an ASL again an ASL? In the first section we will see that, in the case of the polynomial ring, a positive answer to this question was given by A. Conca in [16] . But even in that case, the construction of the new poset and the proof are not at all trivial. We will also recall here the notion of module with straightening laws (MSL), which is a natural generalization of the algebras with straightening laws.

In the second section we will prove that the Veronese modules of the polynomial ring have a structure of MSL. Here, the ASL structure of the polynomial ring given in [16] plays an important role. As a corollary we obtain the result of A. Aramova, S. Bărcănescu, J. Herzog (see [1]) which says that the Veronese modules have a linear resolution. Using the results of W. Bruns from [11] on MSL's, we are able to give an upper bound for the rate of a finitely generated MSL.

In the third section we will look at the Veronese algebra of an ASL. The first step towards proving that it is again an ASL is to construct a new poset. Using the translation of algebraic properties into combinatorial ones, we can sketch the profile of the poset that we want to construct. Unfortunately, we were not able to find a construction that works in general. However, we will prove that the second Veronese algebra of a Hibi ring is again an ASL. The poset that we construct is the second zig-zag poset of the distributive lattice of the Hibi ring.

In the last section of this chapter we construct a new poset starting from a poset of rank three. Then we will prove that it has the combinatorial properties to support an ASL structure of the Veronese algebra.

The weak Lefschetz property (WLP) is an important property of Artinian algebras and it has been recently studied by several authors. The m-times WLP is just a very natural generalization of it. For an overview of the main results achieved so far regarding this topic see [22], [33]. One interesting problem is the description of the Hilbert function of Artinian algebras having the WLP. In [22] the authors give a complete characterization of these Hilbert functions. First they make the remark that if and Artinian algebra has the WLP, then its Hilbert function must be a weak Lefschetz O-sequence in the sense of definition 2.2 and then they construct an Artinian algebra with the WLP for each weak Lefschetz O-sequence.

In Chapter 2 we extend this characterization to Artinian algebras with m-times the WLP and we construct, in a more algebraic fashion, an algebra for each m-times weak Lefschetz O-sequence. We also answer a few natural questions regarding the Betti numbers of these algebras.

At first we will construct, using induction on m, and algebra that will have m-times the WLP. To do this, we need to start with a strongly stable ideal of the polynomial ring in one variable less than we actually need. For the case $m = 1$ we will start from the lex-segment ideal, but the choice of the lex-segment ideal is made only in order to obtain maximal Betti numbers within the class.

The proof of the fact that the algebra we construct has m-times the WLP is based on a slight generalization of the description given by A. Wiebe in [38] of the Artinian algebras with the WLP which are the quotients of the polynomial ring by a strongly stable ideal.

In Section 4 we show first that the algebra we construct has maximal Betti numbers among algebras with a given Hilbert function and m-times the WLP. For this the choice of the lex-segment ideal is needed, but again it is not the only way to obtain such an algebra. In the second part of this section we give a complete description of the Artinian algebras with given Hilbert function, m-times the WLP and maximal Betti numbers within this class. The last part of this section is dedicated to the rigidity property of these algebras. More precisely, if the upper bound is reached by the ith Betti number, then it is reached also by the kth Betti number, for all $k > i$.

In the last section of Chapter 2 we show that, by slightly modifying our construction, we can obtain an ideal whose components of low degree define a radical ideal. To do this we will use some particular type of distraction matrix.

Let I_0 be a monomial ideal of $R = k[x, y]$, with $\dim_k(R/I_0)$ finite. We will study in the third chapter the set of ideals $V(I_0)$ that have I_0 as initial ideal with respect to the degree reverse-lexicographic term order. This set has a natural structure of affine variety, in the sense that an ideal $I \in V(I_0)$ can be considered as a point in the affine space $\mathbb{A}^N$. The coordinates are given by the coefficients of the non-leading terms in the reduced Gröbner basis of the ideal I. These varieties play an important role in the study of Hilbert schemes.

It is known by results of J. Briançon [9] and A. Iarrobino [29] that $V(I_0)$ is actually an affine space, for $I_0 \subset R$. This fact is also a consequence of general results of A. Bialynicki-Birula [4, 5] on smooth varieties with k^*-actions. In the main result of this chapter we give a parametrization of the ideals in $V(I_0)$ when I_0 is a lex-segment. This explicit description of the affine space structure is obtained by associating to each ideal a canonical Hilbert-Burch matrix. We will see that the coordinates of the affine space $\mathbb{A}^N$ will correspond to coefficients of polynomials in $k[y]$. This way we will be able to find also an explicit formula for the dimension of $V(I_0)$.

In Section 3 we will consider ideals of the polynomial ring in three variables, $S = k[x, y, z]$. Let $J_0 \subset S$ be a monomial ideal. It is known that the affine variety $V(J_0)$ is in general not an affine space (see [8], [17]). We will assume that $J_0 = I_0S$, with I_0 a monomial lex-segment ideal of $k[x, y]$, and parametrize the variety $V(J_0)$. This will be an affine space of the same dimension as $V(I_0)$.

In the last section we come to study the Betti strata of $V(J_0)$, with $J_0 \subset S$ as above. We obtain, as predicted by A. Iarrobino in [30], a generalization to codimension two punctual schemes in $\mathbb{P}^2$ of [30, Theorem 2.18]. We will see that $V(J_0)$ is dense in $\mathbb{G}(H)$, where H is the Hilbert series of S/J_0 and $\mathbb{G}(H)$ is the variety that parametrizes graded homogeneous ideals of S for which the Hilbert series of S/I is H.

In the appendix we present a program written in the Computer Algebra system CoCoA [15] which computes parametrizations in the case of polynomial rings in two variables. Many of the results and examples in this thesis were discovered, suggested and double-checked by computer algebra experiments performed with CoCoA.

Contents

Chapter 1

Straightening Laws

The notion of algebra with straightening laws (ASL) was introduced by C. De Concini, D. Eisenbud and C. Procesi in [21]. These algebras give an unified treatment of both algebraic and geometric objects that have a combinatorial nature. The coordinate rings of some classical algebraic varieties (such as determinantal rings and Pfaffian rings) are an example of ASL's. For more details on ASL's consult the book of W. Bruns and U. Vetter [14]. In [10], W. Bruns introduces the concept of module with straightening laws (MSL) over such algebras. One interesting question regarding ASLs is whether their Veronese algebras still have a structure of algebra with straightening laws. In this chapter we will show that the Veronese modules of the polynomial ring $R = K[x_1, \ldots, x_n]$ are MSLs over the Veronese algebra $R^{(d)}$. For small values of n and d, in [26], [27], [28] and [37] T.Hibi and K. Watanabe have some partial results regarding the ASL structure of the Veronese algebra of the polynomial ring. The fact that this algebra is still an ASL was proved for any n and d by A. Conca in [16]. In the second section we will also give an upper bound for the rate of a finitely generated MSL. In the third part we prove in some particular case that the Veronese Algebra of an ASL is still an ASL and we discus the combinatorial problem behind this question. The last section of this chapter is dedicated to a combinatorial construction for posets of rank ≤ 3 that could support an ASL structure of the Veronese algebra.

1.1 Preliminaries

Let us summarize the basic definitions and terminology that we will use later on.

Throughout this chapter we will consider only finite partially ordered sets (posets). Let P be a poset and let $C : \alpha_1 < \ldots < \alpha_t$ be a chain in P, (i.e. a totally ordered subset of P). The *length* of C, will be the cardinality of the set C. A poset is called *pure* if all its maximal chains have the same length.

The *rank* of a poset P, denoted by $rank(P)$ is the supremum of the lengths of all chains contained in P. The *height* of an element $\alpha \in P$, denoted $ht(\alpha)$ is

$$ht(\alpha) = \sup\{\text{length of chains descending from } \alpha\} - 1.$$

Given a natural number $m \geq 1$, a m-*multichain* in P is a weakly increasing sequence of m elements of P: $\alpha_1 \leq \ldots \leq \alpha_m$.
A *poset ideal* of P is a subset I such that if $\alpha \in I, \beta \in P$ and $\beta \leq \alpha$ then $\beta \in I$.

Let k be field, A be a ring and $P \subset A$ be a poset. We call a *monomial* a product of the form $\alpha_1\alpha_2 \ldots \alpha_t$ where $\alpha_i \in P, \ \forall\, i$. A monomial $\alpha_1\alpha_2 \ldots \alpha_t$ is called *standard* if $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_t$. We will use the definition of an ASL which is also used by the author in [10]. This definition is given for graded k-algebras, but one can define an ASL also in the non-graded case (see, for instance, [25]).

Definition 1.1. Let A be a k-algebra, and $P \subset A$ a finite poset. We say that A is a *(graded) algebra with straightening laws* on P over k if the following conditions are satisfied:

(ASL 0) $A = \bigoplus_{i \geq 0} A_i$ is a graded k-algebra such that $A_0 = k$, P consists of homogeneous elements of positive degree and generates A as a k-algebra.

(ASL 1) The set of standard monomials is a basis of A as a k-vector space.

(ASL 2) (Straightening Laws) If α and β are incomparable (written $\alpha \not\sim \beta$) and if

$$\alpha\beta = \sum r_i \gamma_{i1}\gamma_{i2} \ldots \gamma_{it_i}, \tag{1.1}$$

where $0 \neq r_i \in k$ and $\gamma_{i1} \leq \gamma_{i2} \leq \ldots$ is the unique linear combination of standard monomials given by (ASL 1), then $\gamma_{i1} < \alpha$ and $\gamma_{i1} < \beta$ for every i.

When $P \subset A_1$ we say that A is a *homogeneous* ASL over P.

Note that in (1.1) the right-hand side can be equal to 0, but that, even though 1 is a standard monomial, no $\gamma_{i1}\gamma_{i2} \ldots \gamma_{it_i}$ can be 1. These relations are called the *straightening laws* (or *straightening relations*)of A.

An ASL A on P, can be presented as $k[P]/I$, where $k[P]$ is the polynomial ring whose variables are the elements of P and I is the homogeneous ideal generated by the straightening laws. Denote by I_P the monomial ideal of $k[P]$ generated by the incomparable pairs of variables. A linear extension of $(P, <)$ is a total order $<_1$ on P such that $\alpha < \beta$ implies $\alpha <_1 \beta$, for any $\alpha, \beta \in P$. When A is a homogeneous ASL on P and τ is the reverse lexicographic term order with respect to a linear extension of $<$, the polynomials given in (ASL 2) form a Gröbner basis of I and $\mathrm{in}_\tau(I) = I_P$. The algebra $k[P]/I_P$ is an ASL on P and it is called the *discrete* ASL.

The discrete ASL over a poset P can be seen also as the Stanley-Reisner ring of the simplicial complex Δ_P, where Δ_P is the simplicial complex whose vertices are the elements of P and whose facets are the maximal chains of P. This is a useful remark, as it allows one to compute the Hilbert function of any ASL on P by looking at the f-vector of Δ_P.

The following proposition is easy to check, but nevertheless very useful:

Proposition 1.2. *Let A be an ASL on P over k, and $H \subset P$ a poset ideal of P. Then the ideal HA is generated as a k-vector space by the standard monomials containing a factor $\alpha \in H$, and A/HA is an ASL on $P \setminus H$ (where $P \setminus H$ is embedded in A/HA in a natural way).*

This proposition allows one to prove results on ASLs using induction on the cardinality of P. Also the ASL structure in many examples is established this way.

The notion of ASL has a natural generalization to modules in the following sense: For M, a module over an ASL A, we want the generators of M to be partially ordered, a distinguished set of "standard elements" should form a k-basis of M and the multiplication $A \times M \longrightarrow M$ should satisfy a straightening law similar to the straightening law of A. We have the following definition, due to W. Bruns:

Definition 1.3. Let A be an ASL on P over k. And A-module M is called *module with straightening laws* on a finite poset $Q \subset M$ if the following conditions are satisfied:

(MSL 1) For every $x \in Q$ there exists a poset ideal $\mathcal{I}(x) \subset P$ such that the elements

$$\alpha_1\alpha_2\ldots\alpha_i x, \quad \textit{with } \alpha_1 \notin \mathcal{I}(x), \quad \alpha_1 \le \alpha_2 \le \ldots \le \alpha_i, \quad i \ge 0,$$

form a basis of M as a k-vector space. These elements are called *standard elements*.

(MSL 2) For every $x \in Q$ and $\alpha \in \mathcal{I}(x)$ one has

$$\alpha x \in \sum_{y<x} Ay. \tag{1.2}$$

An MSL on a poset Q over a homogeneous ASL A is called *homogeneous* if it is a graded A-module in which Q consists of elements of degree 0.

From (MSL 1) and (MSL 2) it follows immediately, by induction on the rank of x, that each element αx with $\alpha \in \mathcal{I}(x)$ has a standard representation

$$\alpha x = \sum_{y<x} \Big(\sum r_{\alpha x \mu y} \mu \Big) y, \qquad \text{with } 0 \neq r_{\alpha x \mu y} \in k,$$

in which every μy is a standard element.

Remark 1.4. (a) If M is an MSL on a poset Q and if $Q' \subset Q$ is a poset ideal, then the submodule of M generated by Q' is an MSL, too. This allows one to prove theorems on MSLs by noetherian induction on the set of ideals of Q.
(b) In the definition of MSL it would have been enough to require that the standard elements are linearly independent, because (MSL 2) and the induction principle above guarantee that M is generated as a k-vector space by the standard elements.

Given a graded k-algebra $A = \bigoplus_{i\geq 0} A_i$, and $d \geq 2$ a natural number, the *d-Veronese algebra* of A will be: $A^{(d)} = \bigoplus_{i\geq 0} A_{di}$. For every $d \geq 2$ one can consider for every $0 \leq j \leq d-1$ the *jth Veronese module*: $M_j^{(d)} = \bigoplus_{i\geq 0} A_{di+j}$. The module $M_j^{(d)}$ is obviously a $A^{(d)}$ module.

The polynomial ring in n variables $R = K[x_1, \dots, x_n]$ has an ASL structure by taking $x_1, \dots, x_n$ as generators and the order: $x_1 \leq \dots \leq x_n$. In [16], A. Conca proves that the Veronese algebra of the polynomial ring is still an ASL. The monomials in n variables of degree d are a natural choice for the generators of $R^{(d)}$. Unfortunately, already when $n = 2$ and $d = 3$, one cannot give a partial order on $\{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$ in order to obtain an ASL structure for $k[x_1, x_2]^{(3)}$. The main theorem in [28] of T. Hibi shows that neither $k[x_1, x_2, x_3]^{(3)}$ can be given an ASL structure with respect to its semigroup presentation.

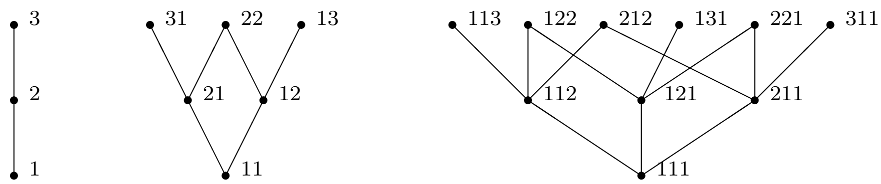
In order to find an ASL structure for $R^{(d)}$ one has to proceed as follows: For $i = 1, \dots, n$ and $j = 1, \dots, d$, take $\ell_{i,j}$ to be generic linear forms such that for any $j_1, \dots, j_n \in \{1, \dots, d\}$ the linear forms $\ell_{1,j_1}, \dots, \ell_{n,j_n}$ are linearly independent.
Take as generators of $R^{(d)}$ all products $\ell_{s_1 1} \dots \ell_{s_d d}$ with the property that $\sum_{i=1}^{i=d} s_i \leq n - d + 1$.
Order these generators as follows: $\ell_{s_1 1} \dots \ell_{s_d d} \leq \ell_{t_1 1} \dots \ell_{t_d d}$ if and only if $s_i \leq t_i$ for every i.

Denote by $H(d) = \{1, \dots, n\}^d$ and order its elements component-wise, i.e. $(\alpha_1, \dots, \alpha_d) \leq (\beta_1, \dots, \beta_d) \iff \alpha_i \leq \beta_i,\ \forall\ i$. Denote by $H_n(d)$ the subposet of $H(d)$ of elements of height $\leq n$, that is

$$H_n(d) = \{(\alpha_1, \dots, \alpha_d) \in H(d) |\ \sum_i \alpha_i \leq n + d - 1\}.$$

The ASL structure of $R^{(d)}$ described above is an ASL structure on $H_n(d)$. For more details see [16].

Here is the Hasse diagram of this poset for $d = 2$ and $d = 3$, when $n = 3$.



P $\qquad$ $P^{(2)} = H_3(2)$ $\qquad$ $P^{(3)} = H_3(3)$

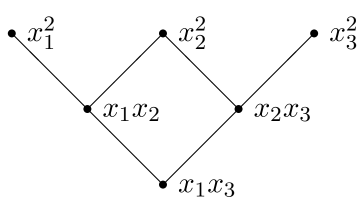
A useful remark is that the 2nd Veronese algebra of the polynomial ring R has a ASL structure also with the usual monomials as generators. If we start with $x_1 < x_2 < \ldots < x_n$, we order the degree two monomials as follows:

$$x_i x_j \leq x_k x_l \iff x_i \leq x_k \text{ and } x_j \geq x_l.$$

The straightening laws will be:
If $x_i x_j \not\leq x_k x_l$, rearrange the indexes i, j, k, l in increasing order, say $i_1 \leq j_1 \leq k_1 \leq l_1$. ($\{i, j, k, l\} = \{i_1, j_1, k_1, l_1\}$ as multi-sets) and define:

$$(x_i x_j)(x_k x_l) = (x_{i_1} x_{l_1})(x_{j_1} x_{k_1}).$$

It is easy to see that $(x_{i_1} x_{l_1})(x_{j_1} x_{k_1})$ is a standard monomial. Also one can easily check that these relations are exactly the relations of the second Veronese algebra. In this case the new poset is $Z_2(P)$ (see Section 3 for definition), but it is also isomorphic to $H_n(2)$.
For example if $n = 3$ the poset will look like this:



and the straightening laws are:

$$\begin{aligned}
(x_1x_2)(x_2x_3) &= (x_1x_3)(x_2^2),\\
(x_1x_2)(x_3^2) &= (x_1x_3)(x_2x_3),\\
(x_2x_3)(x_1^2) &= (x_1x_3)(x_1x_2),\\
(x_i^2)(x_j^2) &= (x_ix_j)^2,
\end{aligned}$$

where $i \neq j$, with $i, j \in \{1, 2, 3\}$.

In [26, Example c)], the authors give an explicit ASL structure over $H_3(3)$ of $k[x, y, z]^{(3)}$.

1.2 The Veronese Module has a MSL Structure over $R^{(d)}$

In this section we will prove that the Veronese modules have a structure of MSLs as $R^{(d)}$-modules.

Let $j \in \{0 \dots d-1\}$ and consider the same generic linear forms that give the ASL structure of $R^{(d)}$. Choose as generators of $M_j^{(d)}$ the products of the form:

$$\ell_{i_1 1} \cdots \ell_{i_j j}, \quad \text{with } i_1 + \ldots + i_j \leq n + j - 1.$$

Order them component-wise, just as in the case of the Veronese algebra of R. So the poset will be in this case $H_n(j)$.

To simplify notation, we will denote the generators of $R^{(d)}$, respectively the generators of $M^{(d)}$ by:

$$\begin{array}{rcllll} f_{\alpha_1 \dots \alpha_d} & := & \ell_{\alpha_1 1} \dots \ell_{\alpha_d d}\ , & \forall\ (\alpha_1, \dots, \alpha_d) & \text{with } \sum_i \alpha_i \leq n + d - 1, \\ g_{i_1 \dots i_j} & := & \ell_{i_1 1} \dots \ell_{i_j j}\ , & \forall\ (i_1, \dots, i_j) & \text{with } \sum_k i_k \leq n + d - 1. \end{array}$$

The fact that the $g_{i_1 \dots i_j}$'s generate $M_j^{(d)}$ follows from the fact that $R^{(d)}$ is generated as an algebra by the $f_{\alpha_1 \dots \alpha_d}$-s for every d.

To every such generator we associate a poset ideal of $H_n(d)$ as follows:

$$\mathcal{I}(g_{i_1 \dots i_j}) = \{f_{\alpha_1 \dots \alpha_d} \mid (\alpha_1, \dots, \alpha_j, \dots, \alpha_d) \not\geq (i_1, \dots, i_j, 1, \dots, 1)\}.$$

It is very easy to see that $\mathcal{I}(g_{i_1 \dots i_j})$ is a poset ideal for any $g_{i_1 \dots i_j}$. We will prove the following:

Theorem 1.5. *Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables. For every $d \geq 2$ and for every $j \in \{0, \dots, d-1\}$, the jth Veronese module $M_j^{(d)}$ is a homogenous MSL on $H_n(j)$ over $R^{(d)}$ with the structure defined above.*

Proof. If $j = 0$ then $M_0^{(d)} = R^{(d)}$ as $R^{(d)}$-modules. In this case, we have a trivial MSL structure over the poset $Q = \{1\}$, with $\mathcal{I}(1) = \phi$ (see [11]). So we will suppose from now on that $j \geq 1$.

In order to prove that we have an MSL structure for $M_j^{(d)}$ we have to check the following:

1. $\forall\ g_{i_1 \dots i_j}$ and $\forall\ f_{\alpha_1 \dots \alpha_d} \in \mathcal{I}(g_{i_1 \dots i_j})$ we have:

$$f_{\alpha_1 \dots \alpha_d} \cdot g_{i_1 \dots i_j} \in \sum_{g_{k_1 \dots k_j} < g_{i_1 \dots i_j}} R^{(d)} \cdot g_{k_1 \dots k_j}.$$

2. The standard elements are linearly independent over k.

To prove 1. let us take $g_{i_1\dots i_j}$ for some $(i_1,\dots,i_j)\in H_n(j)$ and some $f_{\alpha_1\dots\alpha_d}\in\mathcal{I}(g_{i_1\dots i_j})$. This means that $(\alpha_1,\dots,\alpha_d)\not\geq(i_1,\dots,i_j,1,\dots,1)$. So there exists an index $s\in 1,\dots,j$ such that $\alpha_s<i_s$. We have:

$$\begin{aligned} f_{\alpha_1\dots\alpha_d}\cdot g_{i_1\dots i_j} &= \ell_{\alpha_1 1}\dots\ell_{\alpha_d d}\cdot(\ell_{i_1 1}\dots\ell_{i_j j})\\ &= \ell_{\alpha_1 1}\dots\ell_{i_s s}\dots\ell_{\alpha_d d}\cdot(\ell_{i_1 1}\dots\ell_{\alpha_s s}\dots\ell_{i_j j})\\ &= \ell_{\alpha_1 1}\dots\ell_{i_s s}\dots\ell_{\alpha_d d}\cdot g_{i_1\dots,\alpha_s,\dots i_j}. \end{aligned}$$

As $\alpha_s<i_s$ we also have that $g_{i_1\dots,\alpha_s,\dots i_j}<g_{i_1\dots i_j}$. So 1. is true.

As all standard elements are homogeneous polynomials, in order to prove the second part, we only have to look at linear combinations of standard elements of the same degree.
Let F be a linear combination of standard elements of degree $md+j$:

$$F=\sum\lambda\mu g_{i_1\dots i_j},$$

where not all $\lambda\in k$ are zero and every $\mu=f_{\alpha_{11}\dots\alpha_{1d}}\cdot\dots\cdot f_{\alpha_{m1}\dots\alpha_{md}}$ is a standard monomial in $R^{(d)}$ with $f_{\alpha_{11}\dots\alpha_{1d}}\notin\mathcal{I}(g_{i_1\dots i_j})$. In particular $(\alpha_{11},\dots,\alpha_{1j},\dots,\alpha_{1d})\geq(i_1,\dots,i_j,1,\dots,1)$ for all $g_{i_1\dots i_j}$.

If $F=0$ then also $F\cdot\ell_{1j+1}\dots\ell_{1d}=0$. But for all $g_{i_1\dots i_j}$ we have

$$g_{i_1\dots i_j}\cdot\ell_{1j+1}\dots\ell_{1d}=f_{i_1\dots i_j1\dots 1}.$$

As $f_{i_1\dots i_j1\dots 1}\leq f_{\alpha_{11}\dots\alpha_{1d}}\leq\dots\leq f_{\alpha_{m1}\dots\alpha_{md}}$ we have that

$$F\cdot\ell_{1j+1}\dots\ell_{1d}=\sum\lambda f_{i_1\dots i_j1\dots 1}f_{\alpha_{11}\dots\alpha_{1d}}\dots f_{\alpha_{m1}\dots\alpha_{md}}=0$$

is a linear combination of standard monomials in $R^{(d)}$. So, as the standard monomials form a k-basis, all the coefficients λ must be zero.

As $R^{(d)}$ is a homogenous ASL, $M_j^{(d)}$ is a graded $R^{(d)}$-module, and we chose generators of degree 0 for $M_j^{(d)}$, by definition we obtain a homogenous MSL. □

As a consequence of the homogeneous MSL structure, by [13, Theorem 1.1], we have as a corollary the following result, which was proved by A. Aramova, S. Bărcănescu, J. Herzog in [1, (Theorem 2.1)]:

Corollary 1.6. *The $R^{(d)}$-module $M_j^{(d)}$ has a linear resolution for every $j\in\{0,\dots,d-1\}$.*

From [11, (2.6)] we have in general, for any ASL A (not necessarily homogeneous) on a poset P and a MSL M on a poset Q over A, we know there exists a filtration of M:

$$0=M_0\subset M_1\subset\dots\subset M_r=M,$$

with $M_{l+1}/M_l \cong A/A\mathcal{I}(q)$, for some $q \in Q$. The modules M_l are actually the A-modules generated by $q_1, \ldots, q_l$, where $q_1 \leq q_2 \leq \ldots \leq q_r$ are all the elements of Q ordered by a linear extension of the partial order on Q. Using this filtration and the fact that $A\mathcal{I}(q)$ is a MSL over A (see [11, Example 3.1]), we are able to prove the following:

Proposition 1.7. *Let A be an ASL on P over k and M be a MSL on Q over A. Denote by $d = \max\{\deg(p) \mid p \in P\}$ and by $m = \max\{\deg(q) \mid q \in Q\}$. We have:*

$$\beta_{i,j}(M) = 0, \quad \textit{for all } \ i, j \ \textit{ with } \ j - i \geq i(d-1) + m + 1.$$

Where $\beta_{i,j}(M)$ denote the graded Betti numbers of M as an A-module.

Proof. We will prove this using induction on i and on the cardinality of the poset Q. If $i = 0$ everything is clear. We will see in the proof that the case when the cardinality of Q is 1 follows only from induction on i.

Let $i > 0$ and $0 < l < r$. Suppose the assumption holds for $i - 1$ for any poset and for i if the poset has cardinality less than l. In order to make the following exact sequence homogenous, we have to twist $A/A\mathcal{I}(q_l)$ by the degree of q_l:

$$0 \longrightarrow M_{l-1} \longrightarrow M_l \longrightarrow M_l/M_{l-1} \cong A/A\mathcal{I}(q_l)(-\deg(q_l)) \longrightarrow 0.$$

So we obtain the exact sequence:

$$\operatorname{Tor}_i^A(M_{l-1}, k)_j \longrightarrow \operatorname{Tor}_i^A(M_l, k)_j \longrightarrow \operatorname{Tor}_i^A(A/A\mathcal{I}(q_l)(-\deg(q_l)), k)_j.$$

From the short exact sequence

$$0 \longrightarrow A\mathcal{I}(q_l) \longrightarrow A \longrightarrow A/A\mathcal{I}(q_l)(-\deg(q_l)) \longrightarrow 0$$

we obtain that:

$$\operatorname{Tor}_i^A(A/A\mathcal{I}(q_l)(-\deg(q_l)), k)_j = \operatorname{Tor}_{i-1}^A(A\mathcal{I}(q_l)(-\deg(q_l)), k)_j,$$

(this is why the case $\sharp Q = 1$ follows only from induction on i). From [11, Example 3.1] we know that $A\mathcal{I}(q_l)$ is an ASL on the subposet $\mathcal{I}(q_l) \subset P$. So by induction on i we get that $\operatorname{Tor}_i^A(A/A\mathcal{I}(q_l)(-\deg(q_l)), k)_j = 0$, if $j - \deg(q_l) - (i-1) \geq (i-1)(d-1) + d + 1$, which is equivalent to $j - i \geq i(d-1) + \deg(q_l) + 1$, so as $\deg(q_l) \leq m$ we obtain:

$$\operatorname{Tor}_i^A(A/A\mathcal{I}(q_l)(-\deg(q_l)), k)_j = 0, \quad \text{ if } j - i \geq i(d-1) + m + 1.$$

To the left of $\operatorname{Tor}_i^A(M_l, k)_j$, by induction on the cardinality of the poset, we have that:

$$\operatorname{Tor}_i^A(M_{l-1}, k)_j = 0, \quad \text{if } j - i \geq i(d-1) + m + 1$$

and this concludes the proof. □

In [3], J. Backelin introduced, for any homogenous k-algebra A, a numerical invariant called the *rate of* A. This invariant measures how much A deviates from being Koszul. In [1], the authors define a new notion of *rate* for any finitely generated A-module. As $\mathrm{Tor}_i^A(M,k)$ is a finitely generated k-vector space, one can set:

$$t_i(M) := \sup\{j \mid \mathrm{Tor}_i^A(M,k)_j \neq 0\}.$$

Then they define the rate of M as:

$$\mathrm{rate}_A(M) := \sup_{i\geq 1}\{t_i(M)/i\}.$$

Note that $t_i(M)$ is the highest shift in the ith position of the minimal free homogenous resolution of M. With this definition, Proposition 1.7 has the following corollary:

Corollary 1.8. *If M is a MSL over the ASL A, with the above notations we have:*

$$\mathrm{rate}_A(M) \leq d + m.$$

1.3 The Veronese algebra of an ASL

An interesting question about ASLs is whether the Veronese algebra of an ASL is still an ASL. We have seen that so far the only known case is that of the polynomial ring and the complicated structure of its Veronese algebra as an ASL indicates that this question does not have an easy answer.

Let us first see what we should be looking for. Given A an ASL on P over k, we want to find poset $P^{(d)}$ such that $A^{(d)}$ has an ASL structure on $P^{(d)}$ over k. Using the algebraic properties of $A^{(d)}$ and translating them into combinatorial properties of a poset that would support its ASL structure (if there is one), we can outline the properties that a possible $P^{(d)}$ should have. Here are some known facts about ASLs:

1. If A is an ASL on a poset P over k and A is integral then P has an unique minimal element.

2. The Krull dimension of A is equal to the rank of P.

3. The Hilbert function of a homogeneous ASL A on P can be computed directly from the poset P in the following way:

$$\dim_k(A_i) = \sharp\{\text{multichains of length } i \text{ in P}\}.$$

The first property is true because if P would have two different minimal elements, say α and β, then (ASL 2) forces $\alpha\beta = 0$. For a proof of the second property see the book of W. Bruns and U. Vetter [14, (5.10)]. The third remark is the immediate consequence of the fact that the standard monomials (which correspond to the multichains of P) generate A as a k-vector space.

As the Veronese algebra of an integral algebra is again integral, as we know that $\dim_{Krull}(A) = \dim_{Krull}(A^{(d)})$ and as $A_i^{(d)} = A_{di}$ by definition, a possible candidate for $P^{(d)}$ should have the following properties:

1. If P has a unique minimal element, so should $P^{(d)}$.
2. $\text{rank}(P) = \text{rank}(P^{(d)})$.
3. $\#\,\{md\text{-multichains in } P\} = \#\{m\text{-multichains in } P^{(d)}\}$ for all $m \geq 1$.

A poset construction with the above properties that works for every poset is not known to us.

A construction that has properties 2. and 3. is the Zig-Zag poset. Let $P = \{\alpha_1, \ldots, \alpha_n\}$ be a poset. Given $d \geq 2$ a natural number, one can define:

$$Z_d(P) := \{(\alpha_{i_1}, \ldots, \alpha_{i_d}) \mid \alpha_j \in P,\ \forall\, j \text{ and } \alpha_{i_1} \leq \ldots \leq \alpha_{i_d}\}$$

and say that:

$$(\alpha_{i_1}, \ldots, \alpha_{i_d}) \leq (\beta_{i_1}, \ldots, \beta_{i_d}) \iff \begin{array}{rl} & \alpha_{i_1} \leq \beta_{i_1}, \\ \text{and} & \alpha_{i_2} \geq \beta_{i_2}, \\ \text{and} & \alpha_{i_3} \leq \beta_{i_3}, \\ \text{and} & \alpha_{i_4} \geq \beta_{i_4}, \\ & \ldots etc. \end{array}$$

The correspondence between the sets of md-multichains of P and the set of m-multichains in $P^{(d)}$ can be seen easily in the following picture. Suppose $m = 3$ and $d = 4$:

$$\begin{array}{ccccc} \alpha_1 & \leq & \beta_1 & \leq & \gamma_1 \\ \wedge\!\backslash & & \wedge\!\backslash & & \wedge\!\backslash \\ \alpha_2 & \geq & \beta_2 & \geq & \gamma_2 \\ \wedge\!\backslash & & \wedge\!\backslash & & \wedge\!\backslash \\ \alpha_3 & \leq & \beta_3 & \leq & \gamma_3 \\ \wedge\!\backslash & & \wedge\!\backslash & & \wedge\!\backslash \\ \alpha_4 & \geq & \beta_4 & \geq & \gamma_4 \end{array}$$

The md-multichain of P that can be associated to the d-multichain of $Z_d(P)$, $\alpha \leq \beta \leq \gamma$ is: $\alpha_1 \leq \beta_1 \leq \gamma_1 \leq \gamma_2 \leq \beta_2 \ldots \leq \gamma_3 \leq \gamma_4 \leq \beta_4 \leq \alpha_4$. The other way around should also be clear now. So $Z_d(P)$ satisfies 3. It is very easy to see also that it satisfies 2. Unfortunately 1. is almost never satisfied, in the sense that if $d \geq 3$ then we always have at least two minimal elements. The only case in which $Z_d(P)$ satisfies also 1. is when $d = 2$ and P has also a unique maximal element.

Let us first fix some more poset terminology. Let P be a poset and $\alpha, \beta \in P$. Whenever the right-hand side exists, we use the following notation:

$$\alpha \wedge \beta := \sup\{m \in P | m \leq \alpha \text{ and } m \leq \beta\},$$

$$\alpha \vee \beta := \inf\{M \in P | M \geq \alpha \text{ and } M \leq \beta\}.$$

When these elements exist, they are called *greatest lower bound* or *infimum*, respectively *lowest upper bound* or *supremum*.
A poset P in which for any two $\alpha, \beta \in P$, the elements $\alpha \wedge \beta$ and $\alpha \vee \beta$ exist is called a *lattice*.
A lattice P is called *distributive* if the operations defined by $\wedge$ and $\vee$ are distributive to each other. In other words, if for any $\alpha, \beta, \gamma \in P$ we have:

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \quad \text{and}$$
$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma).$$

Another interesting problem related to ASL is to give a description of *integral* posets. We say that a poset P is integral if there exists an ASL on P that is a integral algebra. We have seen that a necessary condition for P is to have a unique minimal element. In [25], T. Hibi shows that every distributive lattice is integral. He constructs for any distributive lattice P an ASL that is integral as an algebra, which is now called the Hibi ring. The generators of this k-algebra are the vertices of the lattice P and the straightening laws are the so called Hibi relations:

$$\alpha\beta = (\alpha \wedge \beta)(\alpha \vee \beta), \quad \forall\ \alpha \not\sim \beta \in P.$$

We will show the following:

Proposition 1.9. *Let P be a distributive lattice and A be the ASL on P given by the Hibi relations. Then $A^{(2)}$ is an ASL over $Z_2(P)$ with the following structure: the vertices of $Z_2(P)$ are the standard monomials of degree 2 in A and the straightening laws are:*

$$(\alpha\beta)(\gamma\delta) = [(\alpha \wedge \gamma)(\beta \vee \delta)][((\alpha \wedge \delta) \vee (\beta \wedge \gamma))((\alpha \vee \delta) \wedge (\beta \vee \gamma))], \quad (1.3)$$

$\forall\ \alpha, \beta, \gamma, \delta \in P,$ with $\alpha \leq \beta,\ \gamma \leq \delta$ and $\alpha\beta \not\sim \gamma\delta$.

In many cases the right hand side can be presented in a shorter form, but this presentation has the advantage to include all cases. For example, if the set $\{\alpha, \beta, \gamma, \delta\}$ is totally ordered, but $(\alpha\beta) \not\sim (\gamma\delta)$, then it is easy to check that (1.3) gives us:

$$(\alpha\beta)(\gamma\delta) = (\alpha_0\delta_0)(\beta_0\gamma_0),$$

where $\alpha_0 \leq \beta_0 \leq c_0 \leq \delta_0$ and $\{\alpha, \beta, \gamma, \delta\} = \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$ as multi-sets. Also if $\alpha \vee \gamma$ and $\beta \wedge \delta$ are comparable, then (1.3) is actually:

$$(\alpha\beta)(\gamma\delta) = [(\alpha \wedge \gamma)(\beta \vee \delta)][(\alpha \vee \gamma)(\beta \wedge \delta)].$$

Proof. We have to check first that the structure described above is an ASL structure and second that this ASL is $A^{(2)}$.

The fact that the standard monomials in $Z_2(P)$ are the standard monomials in P of even degree is clear from the correspondence between m-multichains in $Z_2(P)$ and $2m$-multichains in P. So (ASL 1) is satisfied.

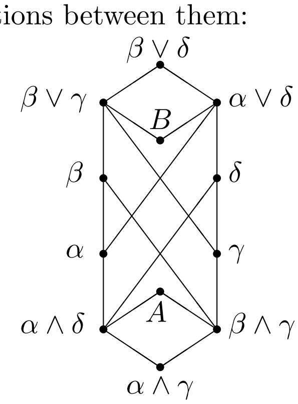
To check (ASL 2) we have to check that:

1. $(\alpha \wedge \gamma)(\beta \vee \delta)$ and $((\alpha \wedge \delta) \vee (\beta \wedge \gamma))((\alpha \vee \delta) \wedge (\beta \vee \gamma))$ are actually vertices in $Z_2(P)$, (that is multichains of length 2 in P),

2. that the right-hand side is a standard monomial in $Z_2(P)$, that is: $(\alpha \wedge \gamma)(\beta \vee \delta) \leq ((\alpha \wedge \delta) \vee (\beta \wedge \gamma))((\alpha \vee \delta) \wedge (\beta \vee \gamma))$,

3. $(\alpha \wedge \gamma)(\beta \vee \delta) \leq (\alpha\beta)$ and $(\alpha \wedge \gamma)(\beta \vee \delta) \leq (\gamma\delta)$.

Here is a picture of the elements of P that we are interested in and the relations between them:



$B = (\beta \vee \gamma) \wedge (\alpha \vee \delta)$

$A = (\alpha \wedge \delta) \vee (\beta \wedge \gamma)$

To check the first point, we will show how this straightening law came up. Suppose that, like in the above picture, $\alpha \not\sim \gamma$ and $\beta \not\sim \delta$. Notice that this is not a restriction, as $\alpha\gamma = (\alpha \wedge \gamma)(\alpha \vee \gamma)$ also when α and γ are comparable. We use the Hibi relations in A to "straighten" $\alpha\gamma$ and $\beta\delta$. It is easy to see that $\alpha \wedge \gamma \leq \beta \vee \delta$. The problem is that $\alpha \vee \gamma$ and $\beta \wedge \delta$ are not always comparable, which means $(\alpha \vee \gamma)(\beta \wedge \delta)$ is not an element of $Z_2(P)$. Suppose they are not comparable. We "straighten" also this product using the Hibi relations. So we get the following:

$$(\alpha \vee \gamma)(\beta \wedge \delta) = [(\alpha \vee \gamma) \wedge (\beta \wedge \delta)][(\alpha \vee \gamma) \vee (\beta \wedge \delta)].$$

Now we just have to show that the first element on the right hand side is A and the second one B. Just by using the distributivity and the fact that

$\alpha \leq \beta$ and $\gamma \leq \delta$ we get:

$$\begin{aligned}(\alpha \vee \gamma) \wedge (\beta \wedge \delta) &= [(\beta \wedge \delta) \wedge \alpha] \vee [(\beta \wedge \delta) \wedge \gamma] \\ &= [\delta \wedge \alpha] \vee [\beta \wedge \gamma] \\ &= A.\end{aligned}$$

$$\begin{aligned}(\alpha \vee \gamma) \vee (\beta \wedge \delta) &= [(\alpha \vee \gamma) \vee \beta] \wedge [(\alpha \vee \gamma) \vee \delta] \\ &= [\gamma \vee \beta] \wedge [\alpha \vee \delta] \\ &= B.\end{aligned}$$

So AB is also a standard monomial and the law that we gave is actually a relation in A.

To prove 2. we just have to look at the drawing and notice that as

$$\alpha \wedge \gamma \leq \alpha \wedge \delta \text{ and } \alpha \wedge \gamma \leq \beta \wedge \gamma,$$

we get that $\alpha \wedge \gamma \leq A$. Using the same way of reasoning we also get that $\beta \vee \delta \geq B$, so 2. holds.
The third point is also immediate.

The straightening laws that we have given can be divided in two types:

- Straightening laws in A, when $\{\alpha, \beta, \gamma, \delta\}$ is not totally ordered,
- Veronese type relations which are 0 when seen as elements of A, when $\{\alpha, \beta, \gamma, \delta\}$ is totally ordered.

As exactly these are also the relations that define $A^{(2)}$, we can conclude that the ASL we have constructed is actually $A^{(2)}$. □

1.4 A poset construction in $\dim \leq 3$

Let P be a poset of rank 3, pure and with unique minimal element μ_0. Also let $d \geq 2$ be a natural number. We will construct a poset $P^{(d)}$ that has the combinatorial properties required in the previous section.

Take as the set of vertices of $P^{(d)}$ the d-multichains in P. Let $\alpha = (\alpha_1, \ldots .\alpha_d)$ and $\beta = (\beta_1, \ldots .\beta_d)$ be two such multichains with $\alpha_1 \leq \ldots \leq \alpha_d$ and $\beta_1 \leq \ldots \leq \beta_d$. For each multichain α we define:

$$v(\alpha) := (ht(\alpha_1), ht(\alpha_2) - ht(\alpha_1), \ldots, ht(\alpha_d) - ht(\alpha_{d-1})).$$

We say that $\alpha \leq \beta$ if the following hold:

1. $\{a_1, \ldots .\alpha_d, \beta_1, \ldots .\beta_d\}$ is totally ordered and
2. $v(\alpha) \leq v(\beta)$ component-wise.

In general, for a vector $v = (v_1, \dots, v_n)$ denote by $|v| := \sum_{i=1}^{i=n} v_i$. In our case, the fact that P has rank 3 implies that for every d-multichain α , $|v(\alpha)| \le 2$. It is easy to see that if $\alpha < \beta$, then $|v(\alpha)| < |v(\beta)|$. Also the only d-multichain α_{min} with $|v(\alpha_{min})| = 0$ is $\alpha_{min} = (\mu_0, \dots, \mu_0)$, and for all other d-multichains β we have $\alpha_{min} \le \beta$.
If $\mathrm{rank}(P) \le 3$, this is a partial order. Antisymmetry and reflexivity are obvious. To check transitivity it is enough to suppose that all inequalities are strict. So let α, β, γ be d-multichains such that $\alpha < \beta$ and $\beta < \gamma$. Then we also have $|v(\alpha)| < |v(\beta)|$ and $|v(\beta)| < |v(\gamma)|$. As $|v(\alpha)|, |v(\beta)|, |v(\gamma)| \in \{0, 1, 2\}$ this implies $|v(\alpha)| = 0$, so $\alpha = \alpha_{min} < \gamma$.
This proof obviously depends on the fact that the rank of P is 3. More than that, this is in general not a partial order for $\mathrm{rank}(P) > 3$.

We have seen that $P^{(d)}$ has a unique minimal element, α_{min}. It is also easy to check that $\mathrm{rank}(P^{(d)}) = \mathrm{rank}(P)$. We actually have $ht(\alpha) = |v(\alpha)|$.

For a natural number m and a poset P, denote $M_m(P) := \{m\text{-multichains in } P\}$. For a m-multichain α in P denote by $\mathrm{supp}_P(\alpha)$ the set of vertices that appear in α. If α is a multichain in $P^{(d)}$ then by $\mathrm{supp}_P(\alpha)$ we denote the set of vertices of P that appear in all the d-multichains that α is made of.

To check that also $\# M_{md}(P) = \# M_m(P^{(d)})$ for all $m \ge 1$ we have to make the following two remarks:

1. If $P_0 = \{0, 1, 2\}$ with the natural order, then $P_0^{(d)} \cong H_3(d)$.

2. There exists a bijection, say:

$$f_{P_0,d} : M_{md}(P_0) \longrightarrow M_m(P_0^{(d)}),$$

such that for any $\alpha \in M_{md}(P_0)$ we have $\mathrm{supp}_{P_0}(\alpha) = \mathrm{supp}_{P_0}(f(\alpha))$.

The isomorphism of posets in (1.) is given by:

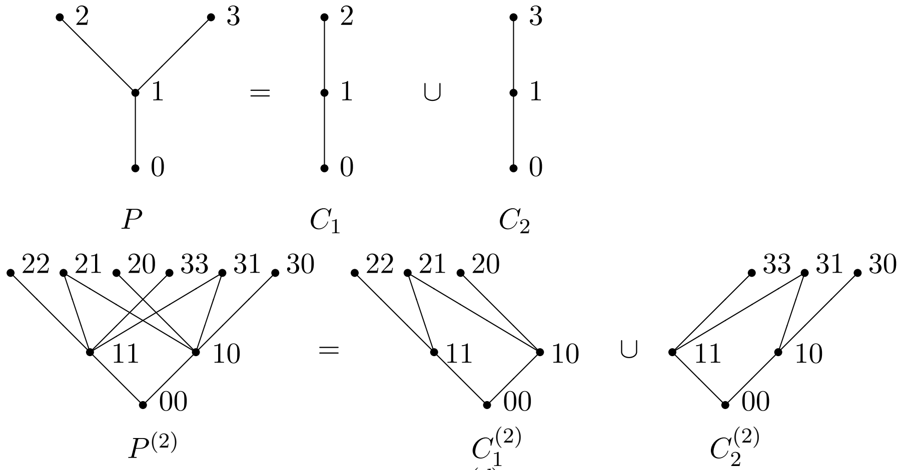
$$\alpha = (\alpha_1, \dots, \alpha_d) \longmapsto v(\alpha) = (\alpha_1, \alpha_2 - \alpha_1, \dots, \alpha_d - \alpha_{d-1}),$$

the inverse of it being:

$$H_3(d) \ni v = (v_1, \dots, v_d) \longmapsto (v_1, v_1 + v_2, \dots, \sum_{i=1}^{i=d} v_i).$$

We know that $M_{md}(P_0)$ and $M_m(P_0^{(d)})$ have the same cardinality for every m. A bijection f as in (2.) exists because for every subposet of $Q \subset P_0$ we have that $Q^{(d)}$ is a subposet of $P_0^{(d)}$. This is again only true for $\mathrm{rank}(P) \le 3$. So we can construct f step by step, starting with $\#(\mathrm{supp}(\alpha)) = 1$.

Any poset P can be seen as the union of its maximal chains. This union is not disjoint, but the construction of $P^{(d)}$ can be done on each such maximal chain C and then $P^{(d)}$ will be the union of $C^{(d)}$.



Now we define $F : M_{md}(P) \longrightarrow M_m(P^{(d)})$ as follows: Take $\alpha \in M_{md}(P)$. We now that there is at least one maximal chain C such that $\operatorname{supp}(\alpha) \subseteq C$. Choose one such C and define:

$$F(\alpha) := f_{C,d}(\alpha) \in C^{(d)} \subset P^{(d)}.$$

From the observations above we can see that $F(\alpha)$ does not depend on the choice of C. If we take another $C' \neq C$ such that $\operatorname{supp}(\alpha) \subseteq C'$, then $\operatorname{supp}(\alpha) \subset C \cap C'$. So $f_{C,d}(\alpha) = f_{C',d}(\alpha) \in C^{(d)} \cap C'^{(d)}$. The function we defined is bijective because it has an inverse $F^{-1} : M_m(P^{(d)}) \longrightarrow M_{md}(P)$ defined by:

$$F^{-1}(\beta) := f^{-1}_{C,d}(\beta) \in C \subset P,$$

where $C \subset P$ is a maximal chain such that $\beta \in C^{(d)}$. The same arguments as above tell us that also F^{-1} is well defined.

Chapter 2

Lefschetz Property

The weak Lefschetz property (WLP) is an important property of Artinian algebras and it has been recently studied by several authors. The m-times WLP is just a very natural generalization of it. For an overview of the main results achieved so far regarding this topic see [22], [33]. One interesting problem is the description of the Hilbert function of Artinian algebras having the WLP. In [22] the authors give a complete characterization of these Hilbert functions. First they make the remark that if and Artinian algebra has the WLP, then its Hilbert function must be a weak Lefschetz O-sequence in the sense of definition 2.2 and then they construct an Artinian algebra with the WLP for each weak Lefschetz O-sequence. In this chapter we extend this characterization to Artinian algebras with m-times the WLP and we construct, in a more algebraic fashion, an algebra for each m-times weak Lefschetz O-sequence. We also answer a few natural questions regarding the Betti numbers of these algebras.

2.1 Preliminaries

Let K be an infinite field of characteristic 0 and let $A = \bigoplus_{d\geq 0} A_d$ be a homogeneous K-algebra, that is an algebra of the form R/I, where R is the polynomial ring in n variables $K[x_1, \dots, x_n]$ and I is a homogeneous ideal. We will denote $h_d(A) = \dim_K(A_d)$ and by $h(A)$ the Hilbert function of A.

Definition 2.1. We say that an Artinian algebra A has the *weak Lefschetz Property (WLP)* if there exists $\ell \in A_1$ such that the multiplication $\times\ell : A_d \longrightarrow A_{d+1}$ has maximal rank for every $d \geq 1$.
Such an element ℓ is called a *weak Lefschetz Element (WLE)* for A.

We say that A has *m-times the weak Lefschetz Property* ($m \in \mathbb{N}$) if there exist $\ell_1, \dots, \ell_m \in A_1$ such that ℓ_1 is a *WLE* for A and ℓ_i is a *WLE* for $A/(\ell_1, \dots, \ell_{i-1}), \forall\, i \in 2, \dots, m$.

The following definition uses the notion of O-sequence, which for us will just mean a sequence of natural numbers that can be the Hilbert function of some graded K-algebra. For more details on O-sequences see [34] or [12, Chapter 4.2].

Definition 2.2. Let $h\ :\ 1 = h_0,\ h_1,\ \dots,\ h_s$ be a finite O-sequence. We say that h is a *weak Lefschetz O-sequence* if :

- h is *unimodal* (i.e. $h_0 < h_1 < \dots < h_k \geq h_{k+1} \geq \dots \geq h_s$ for some $k \in 0, \dots, s$).
- the sequence $1, h_1 - h_0, \dots, h_k - h_{k-1}$ is again an O-sequence.

Inductively, we say that h is a *m-times weak Lefschetz O-sequence* if:

- h is unimodal.
- the sequence $1, h_1 - h_0, \dots, h_k - h_{k-1}$ is a $(m-1)$-times weak Lefschetz O-sequence.

If $I \subset R$ is a monomial ideal, the minimal monomial generating set of I will be denoted by $\mathrm{Gens}(I)$. We will use also the following notion:

Definition 2.3. A monomial ideal $I \subset R$ is called *strongly stable* if: For each monomial $M \in I$ and for each variable x_k that divides M, we have $(\frac{x_i}{x_k})M\ \in\ I,\ \forall\ i < k$.

It is easy to see that in order to verify if a monomial ideal is strongly stable, it is enough to verify the condition above only for the monomials in $\mathrm{Gens}(I)$.

To a homogeneous ideal $I \subset R$ one can canonically attach the *generic initial ideal of* I, $\mathrm{Gin}(I)$ with respect to the reverse lexicographic order. By definition $\mathrm{Gin}(I)$ is the initial ideal of I with respect to the reverse lexicographic order after performing a generic change of coordinates.

For a homogeneous ideal $I \subset R$ one can also define the *lex-segment ideal associated to* I as follows. In general, for a vector space V of forms of degree d, one defines the vector space $\mathrm{Lex}(V)$ to be the vector space generated by the largest $\dim(V)$ forms in lexicographic order. Then, one defines $\mathrm{Lex}(I) := \bigoplus_d \mathrm{Lex}(I_d)$. Macaulay's theorem on Hilbert functions (see for instance [12, 36]) guarantees that $\mathrm{Lex}(I)$ will actually be an ideal, not just a graded vector space. One can immediately notice that the construction of $\mathrm{Lex}(I)$ depends only on the Hilbert function of R/I, so for an O-sequence h we will denote by $\mathrm{Lex}(h)$ the lex-segment ideal for which the Hilbert function of $R/\mathrm{Lex}(h)$ is h.

These ideals play a fundamental role in the investigation of many algebraic, homological, combinatorial and geometric properties of I itself. We recall here some of their properties that we will use later.

When passing from an ideal to its generic initial ideal, the Hilbert function does not change. An important property of the generic initial ideal is that, in characteristic zero, it is strongly stable. The following result shows how the weak Lefschetz property is reflected by the generic initial ideal:

Proposition 2.4. *Let $I \subset R$ be an ideal such that R/I is an Artinian algebra. Then R/I has m-times the weak Lefschetz property if and only if $R/\mathrm{Gin}(I)$ has m-times the weak Lefschetz property.*

This result can be found for $m = 1$ in [38]. To see that it holds for $m > 1$ one just has to follow the same proof and use [18, Lemma 2.1] for m linear forms.

The graded Betti numbers of I, $\mathrm{Gin}(I)$ and $\mathrm{Lex}(I)$ satisfy the following inequalities (for details, see [19]):

Theorem 2.5. (a) $\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathrm{Gin}(I)) \ \forall\ i, j,$

(b) $\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathrm{Lex}(I)) \ \forall\ i, j.$

Recall that a homogeneous ideal I is said to be *componentwise linear* if for all $k \in \mathbb{N}$ the ideal $I_{<k>}$ generated by the elements of degree k in I has a linear resolution.

Also, I is said to be a *Gotzmann* ideal if for all $k \in \mathbb{N}$ the space I_k of forms of degree k in I has the smallest possible span in the next degree according to the Macaulay inequality (see [36, Theorem 3.1]), that is $\dim_K R_1 I_k = \dim_K R_1 \mathrm{Lex}(I_k)$.

Aramova, Herzog and Hibi characterized in [2] the ideals that have the same Betti numbers as their generic initial ideal as follows:

Theorem 2.6. *The following conditions are equivalent:*

(a) $\beta_{ij}(R/I) = \beta_{ij}(R/\mathrm{Gin}(I))$ *,* $\forall\ i, j;$

(b) *I is componentwise linear.*

Ideals with the same Betti numbers as the lex-segment ideal were characterized by Herzog and Hibi in [24]:

Theorem 2.7. *The following conditions are equivalent:*

(a) $\beta_{ij}(R/I) = \beta_{ij}(R/\mathrm{Lex}(I))$ *,* $\forall\ i, j;$

(b) *I is a Gotzmann ideal.*

The fact that if an Artinian algebra has m-times the WLP, then its Hilbert function is a m-times weak Lefschetz O-sequence follows immediately from [22, Remark 3.3]. The unimodality of the Hilbert function is a consequence of the natural grading of the algebra. This guarantees that if $\times \ell_1 : A_j \longrightarrow A_{j+1}$ is surjective then $\times \ell_1 : A_d \longrightarrow A_{d+1}$ is surjective $\forall\ d \geq j$. The second part of the definition of a m-times weak Lefschetz O-sequence is guaranteed by the fact that $A/(\ell)$ in an algebra with $(m-1)$-times the WLP and with Hilbert function $1, h_1 - h_0, \ldots, h_k - h_{k-1}$.

2.2 The construction of $R/\mathcal{W}_m(h)$

Fix $h : 1 = h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$ a m-times weak Lefschetz O-sequence. We will denote by Δh the $(m-1)$-times weak Lefschetz O-sequence: $1,\ h_1 - h_0,\ \ldots,\ h_k - h_{k-1}$. Inductively we will denote $\Delta^1 h = \Delta h$ and by $\Delta^i h$ the $(m-i)$-times weak Lefschetz O-sequence given by $\Delta(\Delta^{i-1}h)$ for $i = 1, \ldots, m$.

For every finite O-sequence $h_0, h_1, \ldots, h_s, 0, 0, \ldots$ with $h_s \neq 0$ we will say that the length of h is s. Returning to our m-times weak Lefschetz O-sequence we will denote by k_i the length of $\Delta^i h$ for every $i = 1, \ldots, m$. Notice that $k = k_1 \geq k_2 \geq \ldots \geq k_m$.

We will construct an ideal $\mathcal{W}_m(h)$ of R such that $R/\mathcal{W}_m(h)$ will be the algebra we are looking for. We will first construct $\mathcal{W}_1(h)$, and then use induction to construct $\mathcal{W}_m(h)$ in the general case.

2.2.1 The case m=1

Let $n = h_1$ and consider I_0 to be the lex-segment ideal of $R' = K[x_1, \ldots, x_{n-1}]$ with Hilbert function Δh. Now we define I_1 to be the ideal $I_0 R$ of R. It is easy to see that the Hilbert function of R/I_1 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots.$$

Also, as $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_0$, we have that $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I_1$. So we know that all the monomials of degree $\geq k+1$ in R that are not in I_1 are divisible by x_n.

In every degree d we will arrange the monomials of R which are not in I_1 in decreasing reverse lexicographic order. Then we will add to the generators of I_1 the largest monomials in each degree such that we obtain the right Hilbert function. But we first have to check how the Hilbert function changes at each step in order to guarantee that this construction can be done.

Let d_0 be the lowest degree in which the Hilbert function of R/I_1 differs from h. As this happens in degree higher than k, we know by the unimodality of h that $h_{d_0}(R/I_1) > h_{d_0}$. So there are "too many" monomials of degree d_0 that are not in I_1.

We define $r_0 = h_{d_0}(R/I_1) - h_{d_0}$. Let T_1 , $\ldots$, T_{r_0} be the largest (in reverse lexicographic order) r_0 monomials of degree d_0 not in I_1. Now we define:

$$I_2 := I_1 + (T_1\ ,\ \ldots\ ,\ T_{r_0}).$$

We want to show that the Hilbert function of R/I_2 is:

$$1 = h_0,\ h_1,\ \ldots,\ h_{d_0-1},\ h_{d_0},\ h_{d_0},\ \ldots,\ h_{d_0},\ \ldots.$$

Obviously the Hilbert function of R/I_2 is equal to the one of R/I_1 in degree smaller than d_0 and now also in degree d_0 it is exactly h_{d_0}.

Denote by $M_{i,1}$, ... , $M_{i,u_i} \in R'$ the monomials of degree i which are not in the original I_0 (u_i will be equal to $h_i - h_{i-1}$). These will be the monomials on degree $\leq k$ in the first $(n-1)$ variables that are not in I_1. So the monomials of degree $d > k$ that are not in I_1 are the following:

$$M_{k,1}x_n^{d-k} \ , \ \ldots \ , \ M_{k,u_k}x_n^{d-k} \ , \ \ldots \ , \ M_{1,1}x_n^{d-1} \ , \ \ldots \ , \ M_{1,u_1}x_n^{d-1}.$$

We have $T_1 = M_{k,1}x_n^{d_0-k}$, and let i_0 and j_0 be the index for which $T_{r_0} = M_{i_0,j_0}x_n^{d_0-i_0}$ (the r_0th largest monomial of degree d_0 not in I_1). After adding to I_1 these first r_0 monomials, we get that $\dim((R/I_2)_d) \leq h_{d_0}$ for $d > d_0$.

Suppose there exists a monomial of degree $d > d_0$, $M_{t,r}x_n^{d-t} \notin I_1$, with $(t < i_0)$ or $(t = i_0$ and $r > j_0)$, i.e. that is not in those first r_0 monomials added to I_1, but $M_{t,r}x_n^{d-t} \in I_2$. As $M_{t,r} \notin I_0$, $M_{t,r}x_n^{d-t}$ must be divisible by a generator of I_2, who itself is divisible by x_n. So it must be divisible by one of $M_{k,1}x_n^{d_0-k}$, ..., $M_{i_0,j_0}x_n^{d_0-i_0}$.

Let $M_{i,j}x_n^{d_0-i}$ be that monomial. It follows that $M_{i,j}|M_{t,r}$, so $i \leq t$. As $i \geq i_0 \geq t$ it follows that $i = t$. So they have the same degree, but $r > j_0 \geq j$ so they are different and the divisibility can not take place - a contradiction. So the only monomials that belong to I_2 but not to I_1 in degree $d \geq d_0$ are exactly $M_{k,1}x_n^{d-k}$, ..., $M_{i_0,j_0}x_n^{d-i_0}$.

So we have shown that after adding the necessary monomials to I_1 in the first degree where this is needed (d_0), the Hilbert function of the new algebra R/I_2 will become:

$$1 = h_0, \ h_1, \ \ldots, \ h_{d_0-1}, \ h_{d_0}, \ h_{d_0}, \ \ldots, \ h_{d_0}, \ \ldots.$$

This procedure can be repeated as from degree $> k$ the original weak Lefschetz O-sequence is decreasing, and after a finite number of steps (at most $s-k$) we will obtain a new ideal, which we will denote by $\mathcal{W}_1(h)$, such that $R/\mathcal{W}_1(h)$ has the desired Hilbert function.

So we have constructed a monomial ideal with Hilbert function h and with the property that $(x_1, \ldots, x_{n-1})^{k+1} \subseteq \mathcal{W}_1(h)$. We also have that all the generators which are divisible by x_n appear in degree $\geq k+1$ and that the generators not divisible by x_n appear in degree $\leq k+1$.

In order to be able to apply induction we will need to prove the following:

Lemma 2.8. *The ideal $\mathcal{W}_1(h)$ is strongly stable.*

Proof. By construction $\mathcal{W}_1(h)$ is a monomial ideal. Let $M \in \text{Gens}(\mathcal{W}_1(h))$ be a monomial of degree d. We want to prove that $x_i\frac{M}{x_j} \in \mathcal{W}_1(h), \forall \ j$ such that $x_j|M$ and $\forall \ i < j$.

We distinguish two cases:
1. If $x_n \not| M$, then M could be seen as a monomial of I_0 which is the lex-segment ideal for Δh. As the lex-segment ideal is strongly stable, we get that $x_i \frac{M}{x_j} \in I_0, \forall\ j$ such that $x_j|M$ and $\forall\ i < j$.
2. If $x_n|M$, then let $j \in 1, \ldots, n$ be such that $x_j|M$ and let $i < j$. Then we have $x_i \frac{M}{x_j} \geq_{rev-lex} M$ and so we must have that $x_i \frac{M}{x_j} \in \mathcal{W}_1(h)$ by construction, because we chose as generators the largest monomials in rev-lex order.

□

2.2.2 The general case

Let $m \in \mathbb{N}$, $m \geq 2$. Assume we can construct an algebra $R'/\mathcal{W}_{m-1}(\Delta h)$, with Hilbert function Δh, such that $\mathcal{W}_{m-1}(\Delta h)$ is a strongly stable ideal of $R' = K[x_1, \ldots, x_{n-1}]$ and that $(x_1, \ldots, x_{n-i})^{k_i+1} \subseteq \mathcal{W}_{m-1}(\Delta h)$ for all $i = 2, \ldots, m-1$.

Now we define $I_1 = \mathcal{W}_{m-1}(\Delta h)R$. The Hilbert function of R/I_1 will be $1 = h_0,\ h_1,\ \ldots,\ h_{k-1},\ h_k,\ h_k,\ \ldots,\ h_k,\ \ldots$ and following the method of adding the needed highest monomials in rev-lex order as in the case of $m = 1$, we can construct an ideal $\mathcal{W}_m(h)$. The same arguments as in the case $m = 1$ prove that the construction can be done, because in that case we didn't use the fact that I_0 was the lex-segment ideal, we just used the fact that it was a strongly stable ideal. The choice of I_0 as the lex-segment ideal is needed for obtaining maximal Betti numbers.

In fact, since $\mathcal{W}_{m-1}(\Delta h)$ is strongly stable, the proof of Lemma 2.1 works also for proving that $\mathcal{W}_m(h)$ is strongly stable.

2.2.3 $R/\mathcal{W}_m(h)$ has m-times the WLP

In this section we will show that the algebra we have constructed so far is actually what we wanted:

Proposition 2.9. *$R/\mathcal{W}_m(h)$ has m-times the weak Lefschetz Property.*

In order to prove this, we will use the following result from [38]:

Lemma 2.10. *If I is a strongly stable ideal of $R = K[x_1, \ldots, x_n]$ then: R/I has the WLP $\Longleftrightarrow x_n$ is a WLE for R/I.*

From this result we can deduce the following one:

Lemma 2.11. *If I is a strongly stable ideal of $R = K[x_1, \ldots, x_n]$, then the following are equivalent.*

1. *R/I has the WLP.*

2. (a) *$h(R/I)$ is unimodal : $h_0 < h_1 < \ldots < h_k \geq h_{k+1} \geq \ldots \geq h_s$,*

(b) $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I$,

(c) *If* $M \in \text{Gens}(I)$ *is divisible by* x_n, *then* $\deg(M) \geq k+1$.

Proof. 1.⇒2. The fact the Hilbert function is unimodal is already known from [22].
By Lemma 3.3 we know that x_n is a WLE for R/I, so the multiplication $\times x_n : (R/I)_d \to (R/I)_{d+1}$ must be of maximal rank, i.e. injective if $d < k$ and surjective if $d \geq k$. This implies immediately that $(x_1, \ldots, x_{n-1})^d \subseteq I$ for $d > k$.
Suppose that there is a minimal generator M of I, which has degree $d < k+1$, and $x_n | M$. Then $\frac{M}{x_n} \neq 0$ in $(R/I)_{d-1}$ but is taken by the multiplication with x_n to $M = 0$ in $(R/I)_d$ - a contradiction with the injectivity of $\times x_n$.

2.⇒1. We will show that x_n is a WLE for R/I. As we have that $(x_1, \ldots, x_{n-1})^{k+1} \subseteq I$, it follows that the multiplication by x_n is surjective in degree $\geq k$.
Let $d < k$ suppose that there exists a monomial of degree d, $M \in R$ and $M \notin I_d$, such that $x_n M \in I_{d+1}$. This means that $x_n M$ is divisible by a minimal generator G of I. As $\deg(G) \leq k$, we have that $x_n \not| G$. This means that $G | M$ contradicting the fact that $M \notin I_d$. □

Let us notice that 2. ⇒ 1. of Proposition 3.4 holds also when I is just a monomial ideal, not necessarily a strongly stable one.

Now we can prove proposition 3.2.

Proof. We will use induction on m:

If $m = 1$ we can easily see that the conditions (a), (b) and (c) from Lemma 3.3 are satisfied by construction so, as $\mathcal{W}_1(h)$ is strongly stable, we can apply Lemma 3.3 and get that $R/\mathcal{W}_1(h)$ has the WLP.

Suppose that the proposition is true for $m-1$. This means that the algebra $R'/\mathcal{W}_{m-1}(\Delta h)$ has $(m-1)$-times the WLP ($R' = K[x_1, \ldots, x_{n-1}]$). As $\mathcal{W}_m(h)$ is strongly stable and again the conditions of Lemma 3.3 are satisfied, we get that $R/\mathcal{W}_m(h)$ has the WLP and, by Lemma 3.2, x_n is a WLE. As by construction $R/\mathcal{W}_m(h) + (x_n) = R'/\mathcal{W}_{m-1}(\Delta h)$ which has by hypothesis $(m-1)$-times the WLP, we get that $R/\mathcal{W}_m(h)$ has m-times the WLP. □

2.3 Ideals with Maximal Betti Numbers

In this section we will first show that $R/\mathcal{W}_m(h)$ has maximal Betti numbers among algebras with Hilbert function h and m-times the WLP. Then we will characterize all other ideals that have maximal Betti numbers within this class. In the third part of this section we will show that these upper bounds are rigid.

2.3.1 $R/\mathcal{W}_m(h)$ has maximal Betti numbers

We want to prove the following:

Proposition 2.12. *For any algebra R/J that has Hilbert function h and m-times the WLP we have:*

$$\beta_{ij}(R/J) \leq \beta_{ij}(R/\mathcal{W}_m(h)), \qquad \forall i, j \geq 0. \tag{2.1}$$

We have seen in Section 2 that for a homogeneous ideal $J \subset R$ taking its generic initial ideal $\mathrm{Gin}(J)$ does not change the Hilbert function, and also that R/J has m-times the WLP if and only if $R/\mathrm{Gin}(J)$ has m-times the WLP. From Theorem 2.3 we have the following inequality:

$$\beta_{ij}(R/J) \leq \beta_{ij}(R/\mathrm{Gin}(J)), \quad \forall i, j \geq 0.$$

So, as $\mathrm{Gin}(J)$ is a strongly stable ideal, it will be enough to prove that (2.1) holds for J strongly stable.

First let us establish some notation. For a monomial $M = x_1^{a_1} \dots x_n^{a_n}$ in $K[x_1, \dots, x_n]$ we define:

$$\max(M) = \max\{i : a_i > 0\}.$$

For a set of monomials $A \subset K[x_1, \dots, x_n]$ and for $i = 1, \dots, n$ we write:

$$m_i(A) = \sharp\{M \in A : max(M) = i\}, \quad m_{\leq i}(A) = \sharp\{M \in A : max(M) \leq i\}.$$

When J is either a vector space generated by monomials of the same degree or a monomial ideal, we set

$$m_i(J) = m_i(G), \qquad m_{\leq i}(J) = m_{\leq i}(G),$$

where G is the set of minimal monomial (vector space or ideal) generators of J. If J is a monomial ideal we will denote by J_i the vector space $\{M \in J : deg(M) = i\}$.

We will need the following result from [19]:

Proposition 2.13. *Let I, J be strongly stable ideals with the same Hilbert function. Assume that $m_{\leq i}(I_j) \leq m_{\leq i}(J_j)$, $\forall i, j \geq 0$. Then one has:*

1. $m_i(J) \leq m_i(I)$, $\qquad \forall\ i > 0$.
2. $\beta_{ij}(R/J) \leq \beta_{ij}(R/I)$, $\qquad \forall\ i, j \geq 0$.

We can now prove Proposition 4.1:

Proof. We saw that we can suppose that J is strongly stable and, as $\mathcal{W}_m(h)$ is also strongly stable, from proposition 4.2 we have that in order to prove that (2.1) holds, we only need to prove that:

$$m_{\leq i}((\mathcal{W}_m(h))_j) \leq m_{\leq i}(J_j), \quad \forall\, i \leq n \text{ and } \forall\, j \geq 0. \tag{2.2}$$

As R/J and $R/\mathcal{W}_m(h)$ have the same Hilbert function it follows immediately that (2.2) holds for $i = n$.

If $i < n$ it is easy to see that, as $\mathcal{W}_{m-1}(\Delta h) = (\mathcal{W}_m(h) + (x_n))/(x_n)$ and if we denote $J_{m-1} = (J + (x_n))/(x_n)$, then we have:

$$m_{\leq i}((\mathcal{W}_m(h))_j) = m_{\leq i}((\mathcal{W}_{m-1}(\Delta h))_j), \quad \forall\, i < n \text{ and}$$

$$m_{\leq i}(J_j) = m_{\leq i}((J_{m-1})_j), \qquad \forall\, i < n.$$

So what we have to prove now is that

$$m_{\leq i}((\mathcal{W}_{m-1}(\Delta h))_j) \leq m_{\leq i}((J_{m-1})_j), \quad \forall\, i < n \text{ and } \forall\, j \geq 0. \tag{2.3}$$

This means that if (2.2) holds for $m - 1$, then it also holds for m. So, in order to conclude, we only need to look at the case $m = 1$.

If $m = 1$ we have:

1. If $j > k_1$ we have by Lemma 3.3 that

$$(x_1, \ldots, x_{n-1})^j \subseteq \mathcal{W}_1(h) \quad \textit{and} \quad (x_1, \ldots, x_{n-1})^j \subseteq J.$$

So, in this case, we actually have equality in (2.3) $\forall\, i < n$.

2. If $j \leq k_1$ By construction $\mathcal{W}_0(\Delta h)$ is the lex-segment ideal and J_0 is still a strongly stable ideal (see [6, Proposition 1.4]). By Lemma 3.2, x_n is a WLE for both R/J and $R/\mathcal{W}_m(h)$, and thus we have that the Hilbert functions of R'/J_0 and $R'/\mathcal{W}_0(\Delta h)$ are equal to Δh.

From the equality of the Hilbert functions we have $|(\mathcal{W}_0(\Delta h))_j| = |(J_0)_j|$ and thus we can apply a result of A.M. Bigatti (see [6, theorem 2.1]) that ensures that (2.2) holds also for $m = 1$.

□

2.3.2 Other Ideals with Maximal Betti Numbers

To simplify notation we introduce, for all $i \in 1, \ldots n$ the following morphism: $\rho_i : K[x_1, \ldots, x_n] \longrightarrow K[x_1, \ldots, x_i]$, with:

$$\rho_i(x_j) = \begin{cases} x_j & \text{if } j \leq i \\ 0 & \text{if } j > i. \end{cases}$$

Notice that if $I \subset K[x_1, \ldots, x_n]$ is a homogeneous ideal, then $\rho_i(I)$ is an ideal of $K[x_1, \ldots, x_i]$. This ideal will have the same generators as the ideal

$I + (x_n, \ldots, x_{n-i+1})/(x_n, \ldots, x_{n-i+1})$.

In this section we will give a description of the ideals J of R such that R/J has Hilbert function h, m-times the WLP and maximal Betti numbers within this category. More precisely we will prove the following:

Proposition 2.14. *Let $J \subset R$ be an ideal such that R/J has Hilbert function h and m-times the weak Lefschetz property ($m \in \mathbb{N}$). The following are equivalent:*

1. *J has maximal Betti numbers among ideals with the above properties.*
2. *J is componentwise linear and the ideal $\rho_{n-m}(\mathrm{Gin}(J))$ is Gotzmann.*

We have already seen in Proposition 4.1 that $\mathcal{W}_m(h)$ has maximal Betti numbers. Let us fix $J \subset R$ as in the hypothesis of Proposition 4.3. From Theorem 2.3 and Proposition 4.1 we get:

$$\beta_{ij}(R/J) \leq \beta_{ij}(R/\mathrm{Gin}(J)) \leq \beta_{ij}(R/\mathcal{W}_m(h)).$$

This means that if R/J has maximal Betti numbers among algebras with m-times the WLP and Hilbert function h, then $\beta_{ij}(R/J) = \beta_{ij}(R/\mathrm{Gin}(J))$. In other words, J must be componentwise linear (by Theorem 2.4.).

Knowing this, we will now concentrate on the properties of $\mathrm{Gin}(J)$. Replacing J with $\mathrm{Gin}(J)$ we may assume that J is strongly stable.

For a homogeneous ideal $J \subseteq R$ and $i \in \mathbb{N}$ we will denote by $J_{\leq i}$ the ideal generated by the elements of J with degree $\leq i$. If J is monomial, then $J_{\leq i}$ will also be monomial.

We already know from Lemma 3.2 that for a strongly stable ideal J, R/J has the WLP if and only if x_n is a WLE for R/J. An easy generalization of this fact is the following:

Lemma 2.15. *Let $J \subseteq R$ be a strongly stable ideal. Then R/J has m-times the WLP $\Longleftrightarrow$ x_{n-i} is a WLE for $R/J + (x_n, \ldots, x_{n-i+1})$ for all $i = 0, \ldots, m-1$.*

Proof. When $m = 1$ the result is just the one of Lemma 3.2.
If $m > 1$ then still we know that x_n is a WLE for R/J. But $R/J + (x_n)$ has $(m-1)$-times the WLP and $J + (x_n)/(x_n)$ will be still strongly stable so we can apply induction. □

We will prove the following result which, together with the above observations, proves Proposition 4.3.

Proposition 2.16. *Let $J \subset R$ be a strongly stable ideal such that R/J has m-times the WLP and Hilbert function h. Then*

$$\beta_{ij}(R/J) = \beta_{ij}(R/\mathcal{W}_m(h)), \forall\ i, j \Longleftrightarrow \rho_{n-m}(J_{\leq k_m}) \text{ is Gotzmann.}$$

Proof. The ideals J and $\mathcal{W}_m(h)$ are strongly stable with the same Hilbert function. We know that $m_{\leq i}(\mathcal{W}_m(h)_j) \leq m_{\leq i}(J_j) \;\; \forall\, i,j$ from the proof of Proposition 4.1. From [19, Proposition 3.7] we know that the following are equivalent:

$$\beta_{ij}(R/J) = \beta_{ij}(R/\mathcal{W}_m(h)),\ \forall\, i,j. \tag{2.4}$$

$$m_{\leq i}(J_j) = m_{\leq i}((\mathcal{W}_m(h))_j),\ \forall\, i,j. \tag{2.5}$$

$\boxed{\Rightarrow}$ So we have $m_{\leq i}(J_j) = m_{\leq i}((\mathcal{W}_m(h))_j),\ \forall\, i,j$, but this means also that

$$m_{\leq i}((\rho_{n-m}(J))_j) = m_{\leq i}((\rho_{n-m}(\mathcal{W}_m(h)))_j) \;\; \forall\, i,j.$$

By construction $\rho_{n-m}(\mathcal{W}_m(h)) = \mathcal{W}_0(\Delta^m h) = \mathrm{Lex}(\Delta^m h)$, which is a strongly stable ideal. Also $\rho_{n-m}(J)$ is a strongly stable ideal because its generators are just the generators of J in the first $n-m$ variables.

By Lemma 4.4 we have that x_{n-i} is a WLE for $R/\rho_{n-i}(\mathcal{W}_m(h))$ and for $R/\rho_{n-i}(J)$, $\forall\, i = 0,\ldots,m-1$. Thus we get that both $R/\rho_{n-m}(\mathcal{W}_m(h))$ and $R/\rho_{n-m}(J)$ have the same Hilbert function, $\Delta^m h$.

So we can apply [19, Proposition 3.7] and obtain that

$$\beta_{i,j}(\rho_{n-m}(J)) = \beta_{i,j}(\mathrm{Lex}(\Delta^m h)),$$

which means by Theorem 2.4. that $\rho_{n-m}(J)$ is a Gotzmann ideal.

$\boxed{\Leftarrow}$ We will show that (2.5) holds.

1. If $i = n-t \geq n-m$ then (2.5) holds from the equality of the Hilbert functions of $R/\rho_{n-t}(J)$ and $R/\rho_{n-t}(\mathcal{W}_m(h))$.

2. If $i < n-m$, we have

$$m_{\leq i}(J_j) = m_{\leq i}((\rho_{n-m}(J))_j),$$

$$m_{\leq i}((\mathcal{W}_m(h))_j) = m_{\leq i}((\rho_{n-m}(\mathcal{W}_m(h))_j).$$

So we only need to prove (2.5) for $\rho_{n-m}(J)$ and $\rho_{n-m}(\mathcal{W}_m(h)) = \mathrm{Lex}(\Delta^m h)$. In this case (2.5) holds because $\rho_{n-m}(J)$ is a Gotzmann ideal, which is equivalent by Theorem 2.5 to the equality of its Betti numbers with the Betti numbers of the lex-segment ideal. This is again equivalent by [19, Proposition 3.7] to

$$m_{\leq i}((\rho_{n-m}(J))_j) = m_{\leq i}((\mathrm{Lex}(\Delta^m h))_j).$$

□

2.3.3 Rigid Resolutions

For a homogenous ideal I it has been shown in [20] that if $\beta_q(I) = \beta_q(\mathrm{Gin}(I))$ then $\beta_i(I) = \beta_i(\mathrm{Gin}(I))$ for all $i \geq q$. This property is called rigidity and it also holds if $\mathrm{Gin}(I)$ is replaced by $\mathrm{Lex}(I)$ or any generic initial ideal of I.

In this section we will prove that algebras with m-times the WLP have a similar property: if one of the Betti numbers reaches the upper bound given by the Betti numbers of $R/\mathcal{W}_m(h)$), then all the following Betti numbers reach it as well. More precisely we will prove that:

Proposition 2.17. *Let $I \subset R$ be a homogeneous ideal such that R/I has m-times the WLP and Hilbert function h. If $\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$ for some q then $\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h))$ for all $i \geq q$.*

Proof. We have already seen that we have the following inequalities:

$$\beta_i(R/I) \leq \beta_i(R/\mathrm{Gin}(I)) \leq \beta_i(R/\mathcal{W}_m(h)).$$

If for some q equality takes place, we have from [20, Corollary 2.4] the following: $\beta_i(R/I) = \beta_i(R/\mathrm{Gin}(I))$ for all $i \geq q$. So we just need to prove that the proposition holds for Gin(I), i.e. we can assume that I is a strongly stable ideal.

From the Eliahou-Kervaire formula for the Betti numbers of stable ideals (see for example [6]) we have that:

$$\beta_i(R/I) = \sum_{s=i}^{n} m_s(I)\binom{s-1}{i-1}. \tag{2.6}$$

In the proof of Proposition 4.1 we have shown that the inequality (2.2) takes place, so from Proposition 4.2 we have that:

$$m_i(I) \leq m_i(\mathcal{W}_m(h)), \qquad \forall\ i > 0. \tag{2.7}$$

So by (2.6) and (2.7) we have that $\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$ also implies the following equality:

$$m_i(I) = m_i(\mathcal{W}_m(h)), \forall\ i \geq q.$$

So, again by (2.6), we get that $\beta_i(R/I) = \beta_i(R/\mathcal{W}_m(h))$ for all $i \geq q$. □

Corollary 2.18. *Let $I \subset R$ be a homogeneous ideal such that the graded algebra R/I has m-times the WLP and Hilbert function h.*

If $\beta_q(R/I) = \beta_q(R/\mathcal{W}_m(h))$ for some q then:

$$\beta_{ij}(R/I) = \beta_{ij}(R/\mathcal{W}_m(h)) \qquad \forall\ i \geq q,\ \forall\ j.$$

Proof. By proposition 4.1 we have $\beta_{ij}(R/I) \leq \beta_{ij}(R/\mathcal{W}_m(h))\ \forall\ i, j$, and as $\beta_i(R/I) = \sum_j \beta_{ij}(R/I)$, Proposition 4.6 implies the desired equality. □

2.4 Ideal of points

In this section we will construct, starting from $\mathcal{W}_m(h)$ and using a distraction matrix, another ideal I (with the same Hilbert function and Betti numbers) such that R/I still has m-times the WLP and $I_{\leq k_1}$ is the ideal of finite set of rational points in $\mathbb{P}_K^{n-1}$.

First let us recall some notions and results that we need. The results on distractions that we will present here were proven by Bigatti, Conca and Robbiano in [7].

Definition 2.19. Let $\mathcal{L} = (L_{ij} \mid i = 1, \ldots, n\ ,\ j\ \in\ \mathbb{N})$ be an infinite matrix with entries $L_{ij} \in R_1$ with the following properties:

1. $\{L_{1j_1}, \ldots, L_{nj_n}\}$ generates R_1 for every $j_1, \ldots, j_n \in \mathbb{N}$.
2. There exists an integer $N \in \mathbb{N}$ such that $L_{ij} = L_{iN}$ for every $j > N$.

We call $\mathcal{L}$ an N*-distraction matrix* or simply a distraction matrix.

Definition 2.20. Let $\mathcal{L}$ be a distraction matrix, and $M = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ a monomial in R. Then the polynomial $D_{\mathcal{L}}(M) = \prod_{i=1}^{n}(\prod_{j=1}^{a_j} L_{ij})$ is called the $\mathcal{L}$*-distraction* of M.

Having defined $D_{\mathcal{L}}(M)$ for every monomial, $D_{\mathcal{L}}$ extends to a K-linear map. Therefore we can consider $D_{\mathcal{L}}(V)$ where V is a subvector space of R, and call it the $\mathcal{L}$*-distraction* of V.

The ideal that we will construct will be $D_{\mathcal{L}}(\mathcal{W}_m(h))$ for some distraction matrix $\mathcal{L}$ with some extra properties. When I is a homogeneous ideal of R , $D_{\mathcal{L}}(I)$ will coincide with $\bigoplus_d D_{\mathcal{L}}(I_d)$, which is in general just a vector space, not an ideal. However, when I is a monomial ideal we have the following result:

Proposition 2.21. *Let $\mathcal{L}$ be a distraction matrix, and $I \subset R$ a monomial ideal.*

1. *The vector space $D_{\mathcal{L}}(I)$ is a homogeneous ideal in R.*
2. *If $M_1, \ldots, M_r$ are monomials in R such that $I = (M_1, \ldots, M_r)$, then we have the following: $D_{\mathcal{L}}(I) = (D_{\mathcal{L}}(M_1), \ldots, D_{\mathcal{L}}(M_r))$.*
3. *$h(R/I) = h(R/D_{\mathcal{L}}(I))$.*
4. *$\beta_{ij}(R/I) = \beta_{ij}(R/D_{\mathcal{L}}(I)) \quad \forall\ i, j$.*

So we know now that $R/D_{\mathcal{L}}(\mathcal{W}_m(h))$ will still have Hilbert function h. But will $R/D_{\mathcal{L}}(\mathcal{W}_m(h))$ still have m-times the weak Lefschetz property? The following result [7, Theorem 4.3] will lead us to the answer of this question:

Theorem 2.22. *Let $\mathcal{L}$ be a distraction matrix and $I \subset R$ be a strongly stable monomial ideal. Then* $\mathrm{Gin}(D_{\mathcal{L}}(I)) = I$.

So from Proposition 2.2 it follows that also $R/D_{\mathcal{L}}(\mathcal{W}_m(h))$ has m-times the weak Lefschetz property.

We still want to show that $D_{\mathcal{L}}(\mathcal{W}_m(h))_{\leq k_1}$ is an ideal of a finite set of points. For this we need to recall the following notion:

Definition 2.23. Let $I = (x_{i_1}^{a_1}, \dots, x_{i_r}^{a_r}) \subset R$ be an irreducible monomial ideal and let $S = \{s = (s_1, \dots, s_r) \mid 1 \leq s_i \leq a_i \ , \forall\ i = 1, \dots, r\}$. Let $\mathcal{L}$ be a distraction matrix, and let V_s be the K-vector space generated by $\{L_{i_1 s_1}, \dots, L_{i_r s_r}\}$. If $V_s \neq V_{s'}$, $\forall\ s, s' \in S$ $(s \neq s')$, we say that $\mathcal{L}$ is *radical for I*.

More generally, if I is any monomial ideal, we say that $\mathcal{L}$ is *radical for I* if $\mathcal{L}$ is radical for all the irreducible components of I.

The following result will show us how we need to choose the distraction matrix $\mathcal{L}$ in order to obtain the desired construction (see [7, Corollary 4.10]).

Proposition 2.24. *Let $I \subset K[x_1, \dots, x_{n-1}]$ be a zero-dimensional strongly stable monomial ideal, and let $\mathcal{L}$ be a distraction matrix which is radical for I, and whose entries are in the polynomial ring $R = K[x_1, \dots, x_n]$.*

Then $D_{\mathcal{L}}(I)$ is the ideal of a finite set of points in $\mathbb{P}_K^{n-1}$ such that $\mathrm{Gin}(D_{\mathcal{L}}(I)) = IR$.

First let us notice that by Proposition 5.3 we have:

$$D_{\mathcal{L}}(\mathcal{W}_m(h))_{\leq k_1} = (D_{\mathcal{L}}(M) \mid M \in \mathrm{Gens}(\mathcal{W}_m(h)), deg(M) \leq k_1).$$

and that by construction the generators of $(\mathcal{W}_m(h))_{\leq k_1}$ are the generators of $\mathcal{W}_{m-1}(\Delta h)$ so they are monomials in $x_1, \dots, x_{n-1}$.
We choose $\mathcal{L}$ to be a distraction matrix such that the first $(n-1)$ lines form a distraction matrix $\mathcal{L}'$ that is radical for $\mathcal{W}_{m-1}(\Delta h)$, and has entries is $K[x_1, \dots, x_n]$. So $D_{\mathcal{L}'}(\mathcal{W}_{m-1}(\Delta h)) = D_{\mathcal{L}}((\mathcal{W}_m(h))_{\leq k_1}) = D_{\mathcal{L}}(\mathcal{W}_m(h))_{\leq k_1}$. Together with the arguments presented so far in this section, this proves the following:

Proposition 2.25. *Let $\mathcal{L}$ be a distraction matrix such that the first $n-1$ lines form a distraction matrix $\mathcal{L}'$ that is radical for $\mathcal{W}_{m-1}(\Delta h)$. Then : $R/D_{\mathcal{L}}(\mathcal{W}_m(h))$ has m-times the WLP, Hilbert function h, the same Betti numbers as $R/\mathcal{W}_m(h)$ and the ideal $D_{\mathcal{L}}(\mathcal{W}_m(h))_{\leq k_1}$ is the ideal of a finite set of rational points in $\mathbb{P}_K^n$.*

This proposition is a generalization of the results obtained by T.Harima, J.C.Migliore, U.Nagel and J.Watanabe in [22, Theorem 3.20].

2.5 Examples

Let $h : 1, 4, 7, 8, 7, 4, 1$ be our given O-sequence and let $R = K[x, y, z, t]$. We will have $\Delta h : 1, 3, 3, 1$ and $\Delta^2 h : 1, 2$. So we see that h is a 2-times weak Lefschetz O-sequence.

We will construct $\mathcal{W}_2(h)$ as well as $\mathcal{W}_1(h)$ and see that they are different.

Let us first construct $\mathcal{W}_2(h)$. We start with the lex-segment ideal of $\Delta^2 h$ which is the ideal

$$\mathrm{Lex}(\Delta^2 h) = (x^2, xy, y^2) \subset K[x, y].$$

The ideal $\mathrm{Lex}(\Delta^2 h)S$, where $S = K[x, y, z]$ will have the Hilbert function:

$$1, 3, 3, 3, 3, \ldots.$$

The monomials of S of degree $d > 2$ that are not in $\mathrm{Lex}(\Delta^2 h)S$ will be:

$$xz^{d-1}, yz^{d-1}, z^d.$$

To obtain the ideal $\mathcal{W}_1(\Delta h)$ we need to add to $\mathrm{Lex}(\Delta^2 h)S$ the first two for $d = 3$ and the third for $d = 4$. So we get:

$$\mathcal{W}_1(\Delta h) = (x^2, xy, y^2, xz^2, yz^2, z^4).$$

Now the next step is considering the ideal $\mathcal{W}_1(\Delta h)R$, which will have Hilbert function:

$$1,\ 4,\ 7,\ 8,\ 8,\ 8, \ldots.$$

The monomials of R of degree $d > 3$ that are not in $\mathcal{W}_1(\Delta h)R$ will be:

$$z^3t^{d-3}, xzt^{d-2}, yzt^{d-2}, z^2t^{d-2}, xt^{d-1}, yt^{d-1}, zt^{d-1}, t^d.$$

So in order to obtain $\mathcal{W}_2(h)$ we need to add the first one for $d = 4$, the next three for $d = 5$, the next three for $d = 6$ and the last one for $d = 7$. So we get that

$$\mathcal{W}_2(h) = (x^2, xy, y^2, xz^2, yz^2, z^4, z^3t, xzt^3, yzt^3, z^2t^3, xt^5, yt^5, zt^5, t^7).$$

To construct the ideal $\mathcal{W}_1(h)$ we start directly with the lex-segment ideal for Δh in $S = K[x, y, z]$:

$$\mathrm{Lex}(\Delta h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4).$$

The ring $R/(\mathrm{Lex}(\Delta h)R)$ will have again the Hilbert function

$$1,\ 4,\ 7,\ 8,\ 8,\ 8,\ \ldots.$$

but the monomials of R of degree $d > 3$ that are not in $\mathrm{Lex}(\Delta h)R$ will be this time:

$$z^3t^{d-3}, y^2t^{d-2}, yzt^{d-2}, z^2t^{d-2}, xt^{d-1}, yt^{d-1}, zt^{d-1}, t^d.$$

And by adding to $\mathrm{Lex}(\Delta h)R$ in the first monomial for $d = 4$, the next three for $d = 5$ etc. we obtain:

$$\mathcal{W}_1(h) = (x^2, xy, xz, y^3, y^2z, yz^2, z^4, z^3t, y^2t^3, yzt^3, z^2t^3, xt^5, yt^5, zt^5, t^7).$$

We will now give an example of a particular distraction and see how it acts on $\mathcal{W}_2(h)$. It is easy to check that the first three lines of the following matrix form a radical distraction for the ideal $\mathcal{W}_1(\Delta h)$ (as needed by Proposition 5.7):

$$\mathcal{L} = \begin{pmatrix} x & x-t & x-2t & x-3t & x-3t & \dots \\ y & y-t & y-2t & y-3t & y-3t & \dots \\ z & z-t & z-2t & z-3t & z-3t & \dots \\ t & t & t & t & t & \dots \end{pmatrix}.$$

As the highest degree of the generators in first three variables is 4, we can consider $L_{ij} = L_{i4}, \forall\ j \geq 4$. The ideal $D_{\mathcal{L}}(\mathcal{W}_2(h))_{\leq 3}$ will be:

$$D_{\mathcal{L}}(\mathcal{W}_2(h))_{\leq 3} = (x(x-t),\ xy,\ y(y-t),\ xz(z-t)).$$

One can check easily that this ideal is radical.

Let us consider also the following ideal:

$$I = (x^2, y^2, z^2, xyzt, xyt^3, xzt^3, yzt^3, xt^5, yt^5, zt^5, t^7).$$

By Lemma 3.4 we see immediately that R/I has the WLP.

In order to have a more general picture of the Betti numbers of algebras with Hilbert function h, we will also look at $R/\mathrm{Lex}(h)$. This algebra will have the highest Betti numbers possible in this case.

$\mathrm{Lex}(h) = (x^2,\ xy,\ xz,\ xt^2,\ y^3,\ y^2z,\ y^2t^2,\ yz^3,\ yz^2t,\ yzt^3, yt^4,\ z^5,\ z^4t,$
$z^3t^3,\ z^2t^4,\ zt^5,\ t^7).$

Now let's take a look at the Betti diagrams of the ideals constructed so far (the Betti diagram of $D_{\mathcal{L}}(\mathcal{W}_2(h))$ is equal to the one of $\mathcal{W}_2(h)$).

	1	2	3	4
1	3	3	1	-
2	3	6	4	2
3	3	8	7	2
4	4	11	10	3
5	3	9	9	3
6	1	3	3	1

$R/\mathrm{Lex}(h)$

	1	2	3	4
1	3	3	1	-
2	3	5	2	-
3	2	5	4	1
4	3	9	9	3
5	3	9	9	3
6	1	3	3	1

$R/\mathcal{W}_1(h)$

	1	2	3	4
1	3	2	-	-
2	2	4	2	-
3	2	5	4	1
4	3	9	9	3
5	3	9	9	3
6	1	3	3	1

$R/\mathcal{W}_2(h)$

	1	2	3	4
1	3	-	-	-
2	-	3	-	-
3	1	3	4	1
4	3	9	9	3
5	3	9	9	3
6	1	3	3	1

R/I

We can notice that $R/\mathrm{Lex}(h)$ has the largest Betti numbers. Just as predicted, $R/\mathcal{W}_1(h)$ has larger Betti numbers than $R/\mathcal{W}_2(h)$and R/I. We can also notice that the inequality is strict in some cases. The fact that the Betti numbers of $R/\mathcal{W}_2(h)$ are all larger than the ones of R/I is just a coincidence.

Chapter 3

Parametrizations of Ideals in $k[x, y]$

For a filed k of any characteristic, a monomial ideal $I_0 \subset K[x_1, \ldots, x_n]$ and any term order τ, the set $V_h(I_0) = \{I \subset K[x_1, \ldots, x_n] \mid I$ homogeneous, with $\mathrm{in}_\tau(I) = I_0\}$ has a natural structure of affine variety. If we have that $\dim_k(K[x_1, \ldots, x_n]/I_0) < \infty$, also the set in which we consider all ideals (not necessarily homogeneous), $V(I_0) := \{I \subset K[x_1, \ldots, x_n] \mid \mathrm{in}(I) = I_0\}$ has a structure of affine variety.
The main goal of this chapter is to parametrize the affine variety $V(I_0)$, when I_0 is a monomial, lex-segment ideal of $R = k[x, y]$, τ is the degree reverse-lexicographic (DRL) term order, and $\dim_k(R/I_0) < \infty$. It is known by results of J. Briançon [9] and A. Iarrobino [29] that $V(I_0)$ is an affine space. This fact is also a consequence of general results of A. Bialynicki-Birula [4, 5]. The parametrization that we will find associates to each ideal $I \in V(I_0)$ a canonical Hilbert-Burch matrix. This will also allow us to find a formula for the dimension of $V(I_0)$. We will also extend somehow this results to homogeneous ideals of the polynomial ring $k[x, y, z]$, with initial ideal generated only in two variables. This will allow us to study also the Betti strata of $V(J_0)$, where $J_0 = I_0 k[x, y, z]$ and I_0 is a monomial, lex-segment ideal of $R = k[x, y]$.

3.1 Preliminaries

All the initial terms and ideals that we will consider from now on will be with respect to the degree reverse lexicographic term order.

Let $I_0 \subset R$ be a monomial ideal as above:

$$I_0 := (x^t, x^{t-1}y^{m_1}, \ldots, xy^{m_{t-1}}, y^{m_t}).$$

Notice that we have $0 = m_0 \le m_1 \le \ldots \le m_t$; and let us define $d_i := m_i - m_{i-1}$ for all $i = 1, \ldots t$. It is clear that the ideal I_0 is uniquely determined

by the sequence of the m_i's, so also by that of the d_i's.
Now define the following matrix:

$$X = \begin{pmatrix} y^{d_1} & 0 & \dots & 0 \\ -x & y^{d_2} & \dots & 0 \\ 0 & -x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & y^{d_t} \\ 0 & 0 & \dots & -x \end{pmatrix}$$

and let A be another $(t+1) \times t$ matrix, with entries in the polynomial ring in one variable $k[y]$, with the following property:

$$\deg(a_{i,j}) \leq \begin{cases} \text{Min}\{i - j + m_j - m_{i-1} - 1, \quad d_i - 1\} & \text{if } i \leq j, \\ \text{Min}\{i - j + m_j - m_{i-1} \quad\quad , \quad d_j - 1\} & \text{if } i > j, \end{cases} \tag{3.1}$$

where $i = 1, \dots, t+1$ and $j = 1, \dots, t$. We will denote by $\mathscr{A}_{I_0}$ the set of all matrices that satisfy the above condition. Notice that $\mathscr{A}_{I_0} = \mathbb{A}^N$, where N is the sum over i and j of the above bounds $+1$, whenever these bounds are positive. At the end of the next section we will compute the exact formula for N, when I_0 is a lex-segment ideal.

For $i = 1, \dots, t+1$ and any $A \in \mathscr{A}_{I_0}$, denote by $[X + A]_i$ the matrix obtained by deleting the ith row of the matrix $X+A$. We define the following polynomials for $i := 0, \dots, t$:

$$f_i := (-1)^i \det([X + A]_{i+1}).$$

Let $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ be the application defined by:

$$\psi(A) := I_t(X + A),$$

where by $I_t(X + A)$ is the ideal generated by t-minors of the matrix $X + A$. In particular $\psi(A)$ is the ideal generated by $f_0, \dots, f_t$.

3.2 Main theorem

Theorem 3.1. *Let $I_0 \subset R = k[x,y]$ be a monomial lex-segment ideal with $\dim(R/I_0) < \infty$. Then, the application $\psi : \mathscr{A}_{I_0} \longrightarrow V(I_0)$ is bijective.*

This will be the parametrization of $V(I_0)$ that we are looking for. To prove this we have to prove three things:

1. The application ψ is well defined.

2. The application ψ is injective.

3. The application ψ is surjective.

We believe the result to be true without the assumption that I_0 is a lex-segment ideal. But we were not able to prove the second point without this hypothesis. Hoping that such a proof exists, we present here proofs of the first and the third point that work for any monomial ideal I_0 with $\dim(R/I_0) < \infty$.

3.2.1 Proof of 1

We want to prove that

$$\mathrm{in}(I_t(X+A)) = I_0, \text{ for all } A \in \mathscr{A}_{I_0}.$$

First we show that $\mathrm{in}(f_i) = x^{t-i}y^{m_i}$. Then we will show that $\{f_0, \ldots, f_t\}$ form a Gröbner basis.

Notice that the conditions imposed on the degrees of the entries in A we get that:

$$\begin{array}{rcc} \deg(a_{i,i}) & \leq & d_i - 1, \\ \deg(a_{i+1,i}) & \leq & \mathrm{Min}(1, d_i - 1). \end{array}$$

The product on the diagonal of $[X+A]_{i+1}$ will be:

$$(y^{d_1} + a_{1,1}) \ldots (y^{d_i} + a_{i,i})(-x + a_{i+2,i+1}) \ldots (-x + a_{t+1,t})$$

and by the above observation it follows that its leading term will be $x^{t-i}y^{m_i}$.

Now let us look at the other monomials that may appear in the support of f_i. Let $\sigma : \{1, \ldots, t\} \longrightarrow \{1, \ldots, \widehat{i+1}, \ldots, t+1\}$ be a bijection. Denote:

$$\widetilde{\sigma}(j) = \begin{cases} \sigma(j) & \text{if } \sigma(j) < i+1, \\ \sigma(j) - 1 & \text{if } \sigma(j) > i+1. \end{cases}$$

and for $j = 1, \ldots, t$ denote:

$$\alpha_{\sigma(j),j} := \begin{cases} y^{d_j} + a_{\sigma(j),j}, & \text{if } \sigma(j) = j, \\ -x + a_{\sigma(j),j}, & \text{if } \sigma(j) = j+1, \\ a_{\sigma(j),j}, & \text{otherwise.} \end{cases}$$

So the other monomials that may appear in the support of f_i will appear from products of the form:

$$\prod_{j=1}^{t} \alpha_{\sigma(j),j}.$$

By construction we have that for any bijection σ :

$$\deg(\prod_{j=1}^{t} \alpha_{\sigma(j),j}) \leq \underbrace{\sum_{j=1}^{t} \sigma(j) - j + m_j - m_{\sigma(j)-1}}_{\substack{|| \\ t-i+m_i \\ || \\ \deg(x^{t-i}y^{m_i}).}}$$

First let us notice that if for some j we have $\sigma(j)-j+m_j-m_{\sigma(j)-1} < 0$, then this actually means that the whole product is 0. By the above inequality we get that all the monomials in the support of f_i that are not divisible by a power of x higher than or equal to $t-i$ are smaller than $x^{t-i}y^{m_i}$ in degree lexicographic order.

In order to have that $\mathrm{in}(f_i) = x^{t-i}y^{m_i}$ we have to check that a monomial, which is divisible by a power of x higher than $t-i$, has degree strictly less than $t-i+m_i$.

To check this, we first have to notice that to obtain such a monomial, we have to take an $\alpha_{\sigma(j),j}$ that has $\widetilde{\sigma}(j) = j+1 > j$ (i.e. $\alpha_{\sigma(j),j}$ is an element of the matrix A_i that lies just under the diagonal). As σ is a bijection this implies that we also have to have, for some k, that $\widetilde{\sigma}(k) < k$ (i.e $\alpha_{\sigma(k),k}$ lies above the diagonal). So, by definition, we get that: $\deg(\alpha_{\sigma(k),k}) \leq \sigma(k) - k + m_k - m_{\sigma(k)-1} - \mathbf{1}$.

As the syzygy module of I_0 is generated by the columns of the matrix X, by an optimization of the Buchberger algorithm (see [31], Remark 2.5.6), to show that $\{f_0, \ldots, f_t\}$ are a Gröbner basis we only have to look at the S-polynomials of the form:

$$y^{d_i} f_{i-1} - x f_i, \quad \text{for all } i = 1, \ldots, t,$$

and check that they can be written as $\sum_{j=0}^{t} \beta_j f_j$, with:

$$\mathrm{in}(\beta_j f_j) \leq \mathrm{in}(y^{d_i} f_{i-1} - x f_i), \text{for all } j = 0, \ldots, t.$$

But by construction we have that:

$$y^{d_i} f_{i-1} - x f_i + \sum_{j=0}^{t} a_{j+1,i} f_j = 0.$$

As all the $a_{i,j}$ are polynomials in $k[y]$ and all the leading terms of the f_j's are divisible by different powers of x, we get that the leading terms of the $a_{i,j} f_j$'s cannot cancel each other, so we must have:

$$\mathrm{Max}_j \{\mathrm{in}(a_{i,j} f_j) | a_{i,j} \neq 0\} = \mathrm{in}(y^{d_i} f_{i-1} - x f_i).$$

This ends the proof of the fact that ψ is well defined.

3.2.2 Proof of 2

We will use in this proof the fact that I_0 is a lex-segment ideal, i.e. for any monomial $u \in I_0$ of degree d, all the monomials v of degree d, with $v >_{Lex} u$ are also in I_0.

It is easy to check that I_0 is a lex-segment ideal iff $d_i > 0,\ \forall\, i = 1, \ldots, t$. In this case, we can give a more accurate description of maximal degrees that may appear in $X + A$.

First, above the diagonal $(i \leq j)$ we will have

$$\mathrm{Min}\{i - j - 1 + m_j - m_{i-1}, d_i - 1\} = d_i - 1,$$

because

$$\begin{aligned} i - j - 1 + m_j - m_{i-1} &= i - j - 1 + m_j - m_{j-1} + \ldots + m_i - m_{i-1} \\ &= i - j + d_j + d_{j-1} + \ldots + d_{i+1} + d_i - 1 \\ &\geq 0 + d_i - 1. \end{aligned}$$

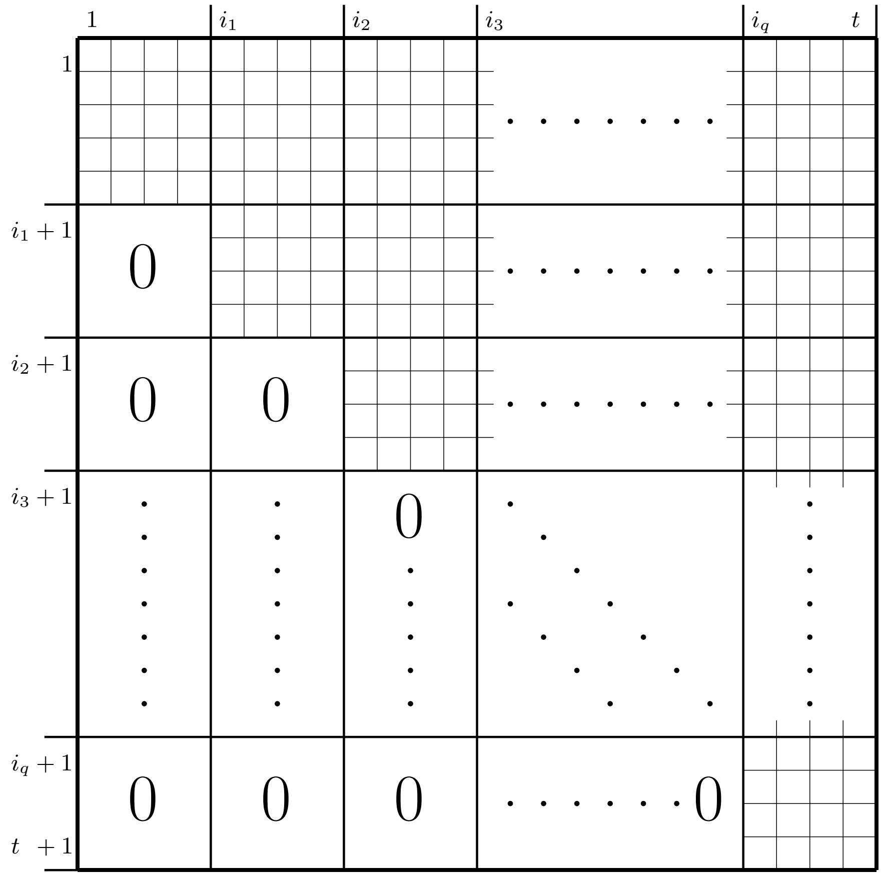
Below the diagonal $(i > j)$ things are slightly more complicated:

$$\begin{aligned} i - j + m_j - m_{i-1} &= i - j + m_j - m_{j+1} + \ldots + m_{i-2} - m_{i-1} \\ &= i - j - d_{j+1} - \ldots - d_{i-1} \\ &\leq 1. \end{aligned}$$

Let us fix two sets of "special" indices:

$$\begin{aligned} \mathcal{J} &:= \{j \in 1, \ldots, t \mid d_j \geq 2\}, \\ \mathcal{I} &:= \{i \in 1, \ldots, t \mid d_i \geq 3\} \cup \{1\}. \end{aligned}$$

Notice that $\mathcal{I} \subseteq \mathcal{J}$. Let $1 = i_0, i_1 \ldots, i_q$ the elements of $\mathcal{I}$ in increasing order. By the above observations, we can divide the matrix $X + A$ into blocks depending on the indices in $\mathcal{I}$ as follows:



So, except for the last $(t - i_q)$ columns, the entries in each column are, from some point on, equal to 0.

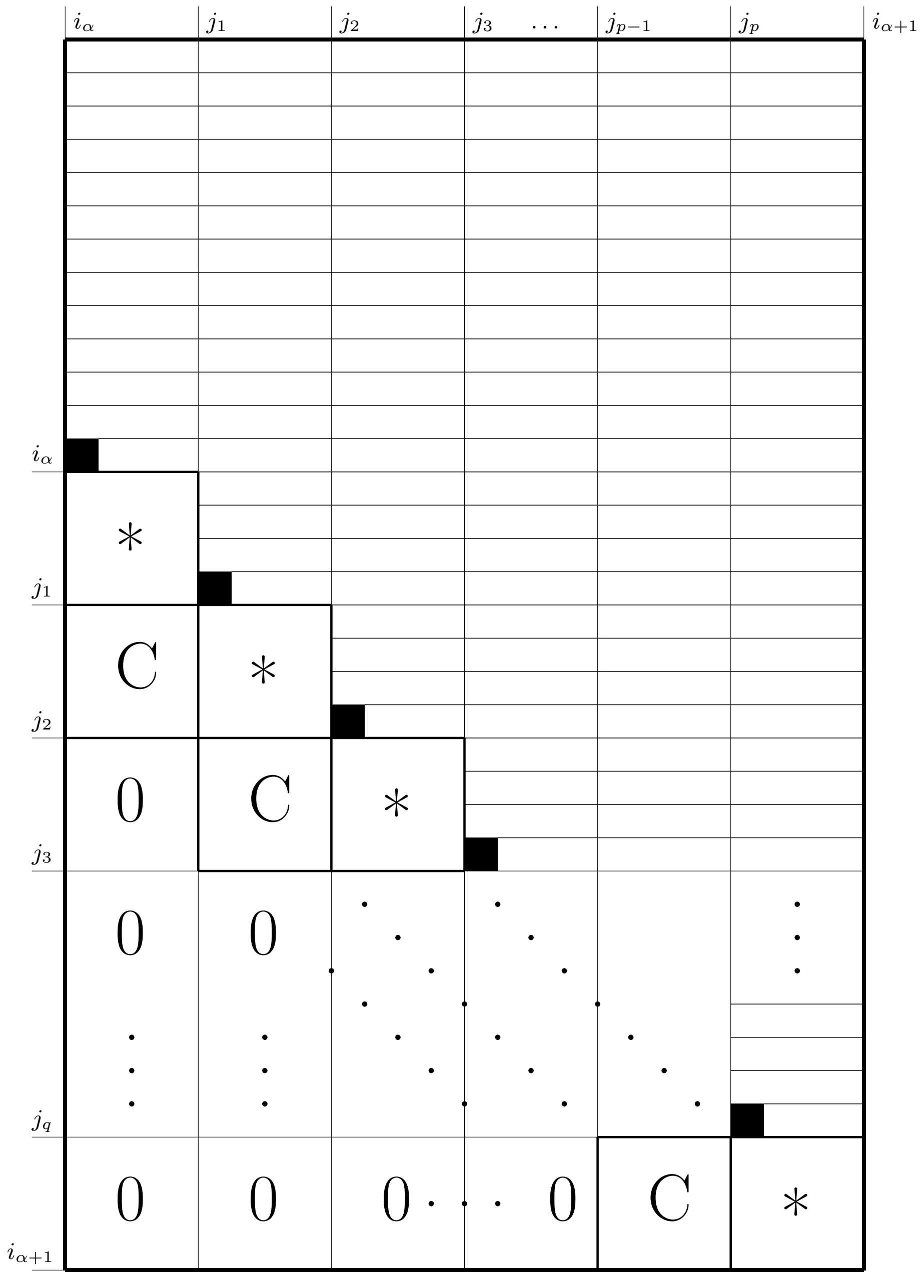
Let $\alpha \in 1, \dots, q$. And let $\{j_1, \dots, j_p\} := \{j \in \mathcal{J} \mid i_\alpha < j < i_{\alpha+1}\}$. We will take a look at the first $i_{\alpha+1}$ rows of the columns indexed from i_α to $i_{\alpha+1} - 1$.

For $s \geq 0$ we denote by $\cdot y^s$, a polynomial in $k[y]$, with degree $\leq s$. In the next figure the part above the diagonal of kth row consists only of $\cdot y^{d_k - 1}$. In particular, if $k \notin \mathcal{J}$, then it is made of constants. The blocks denoted by C consist also only of constants. The black squares, which correspond to elements on the diagonal in the position (j, j) with $j \in \mathcal{J}$ are $y^{d_j} + \cdot y^{d_j - 1}$.

And finally, all matrices denoted by $\ast$ are of the form:

$$\ast = \begin{pmatrix} -x + \cdot y & y & c & \dots & c \\ \cdot y & -x + c & y & \dots & c \\ \cdot y & c & -x + c & \dots & c \\ \vdots & \vdots & \vdots & \ddots & y \\ \cdot y & c & c & \dots & -x + c \end{pmatrix}.$$

For simplicity, we denote by c a constant in general, so all c that appear can be different one from the other, and can also be zero.



Now let us suppose that there exist $A, B \in \mathscr{A}_{I_0}$ such that $I_t(X+A) = I_t(X+B) := I$. We want to prove that in this case $A = B$. For $i = 0, \dots, t$ denote:

$$f_i := (-1)^i \det([X+A]_{i+1}),$$
$$g_i := (-1)^i \det([X+B]_{i+1}).$$

We will prove that $f_i = g_i$, $\forall\ i = 0, \dots, t$. This implies that $A = B$ because the columns of the matrix $(X+A)$ and $(X+B)$ are syzygies for the f_i's, respectively the g_i's. So, if $f_i = g_i$ for all i, then also the columns of $(X+A) - (X+B) = A - B$ will be again syzygies. But the entries of $A - B$ are polynomials in $k[y]$, and as the leading terms of the f_i's involve different powers of x, they must all be zero. So $A = B$.

To prove that $f_i = g_i$, $\forall\ i = 0, \dots, t$, we will first make two important observations.

As in the proof of 1, we denote by $\alpha_{i,j}$ the entries of $X + A$ and with $\beta_{i,j}$ the entries of $X + B$. These are of the following form:

$$\alpha_{i,j} := \begin{cases} y^{d_i} + a_{i,i} & \text{if } i = j, \\ -x + a_{i+1,i} & \text{if } i = j+1, \\ a_{i,j} & \text{otherwise.} \end{cases}$$

and

$$\beta_{i,j} := \begin{cases} y^{d_i} + b_{i,i} & \text{if } i = j, \\ -x + b_{i+1,i} & \text{if } i = j+1, \\ b_{i,j} & \text{otherwise.} \end{cases}$$

First we will show that the homogeneous component of maximal degree of f_i is equal to he homogeneous component of maximal degree of g_i, for all $i = 0, \dots, t$.

Let ω be the weight vector $(1,1)$. For a polynomial $f \in k[x,y]$ we denote by $\mathrm{in}_\omega(f)$ the sum of the monomials of maximal degree. For an ideal $I \subset k[x,y]$ we denote by $\mathrm{in}_\omega(I) := \langle \mathrm{in}_\omega(f) \mid f \in I \rangle$. We will prove the following lemma:

Lemma 3.2. *Let $I_0 \subset R$ be a monomial lex-segment ideal. Let $A, B \in \mathscr{A}_{I_0}$ be two matrices such that $I_t(X+A) = I_t(X+B)$. Then, with the above notations, we have:*

$$\mathrm{in}_\omega(f_i) = \mathrm{in}_\omega(g_i), \quad \forall\ i = 0, \dots, t.$$

Proof. As the DRL order is a refinement of the partial order given by the weight vector ω, we have by [35] that:

$$\mathrm{in}(\mathrm{in}_\omega(I)) = \mathrm{in}(I) = I_0.$$

By the proof of 1. we know that both $\{f_i\}_{i=0,\dots,t}$ and $\{g_i\}_{i=0,\dots,t}$ are Gröbner bases with respect to the DRL term order. So again by [35] we get that $\{\mathrm{in}_\omega(f_i)\}_{i=0,\dots,t}$ and $\{\mathrm{in}_\omega(g_i)\}_{i=0,\dots,t}$ are DRL Gröbner bases.

Again, by looking at the proof of 1. we see that the monomials of maximal degree of the f_i's appear in the following way: they will be the maximal degree part of the products

$$\prod_{j=1}^{t} \alpha_{\sigma(j),j}$$

with the property that $\deg(\alpha_{\sigma(j),j}) = \sigma(j) - j + m_j - m_{\sigma(j)-1}$ for all j. In other words, the polynomials $\mathrm{in}_\omega(f_0), \dots, \mathrm{in}_\omega(f_t)$ will be the maximal minors of a matrix $X + A'$, where the entries $a'_{i,j}$ of A' have the following property:

$$a'_{i,j} = \begin{cases} c_{i,j} y^{i-j+m_j-m_{i-1}} & \text{if } j < i \text{ and } 0 \le i-j+m_j-m_{i-1} < d_j, \\ 0 & \text{otherwise,} \end{cases}$$

with $c_{i,j} \in k$.

The same holds for the polynomials $\mathrm{in}_\omega(g_0), \dots, \mathrm{in}_\omega(g_t)$. Let us say they are the maximal minors of a matrix $X+B'$, with B' having the same property as A'.

As all these polynomials are homogeneous, their leading term is the same for the DRL and the Lex term order. So we find ourselves in the case already solved by Conca and Valla in [17]. That is the matrices A' and B' parametrize the same homogeneous ideal, $\mathrm{in}_\omega(I)$. So, by [17, Theorem 3.3], they must be equal. Thus we also have that $\mathrm{in}_\omega(f_i) = \mathrm{in}_\omega(g_i)$ for all $i = 0, \dots, t$. □

Second we will prove the following lemma:

Lemma 3.3. *Let $f \in I$ be a polynomial such that x^t does not divide any monomial $m \in \mathrm{Supp}(f)$. Then f can be written as:*

$$f = \sum_{i=1}^{t} a_i f_i,$$

with $a_i \in k[y]$ and $\deg(f_i) \le \deg(f)$.

Proof. We have that $\mathrm{in}(f) = x^s y^r$ with $s < t$. As $\mathrm{in}(f) \in \mathrm{in}(I)$, we have that $r \ge m_{t-s}$. We now define a new polynomial:

$$f' := f - \mathrm{LC}(f) y^{r-m_{t-s}} f_s,$$

where $\mathrm{LC}(f)$ is the leading coefficient of f. By construction, the monomials that appear in the support of the f_i's are not divisible by x^t for $i = 1, \dots, t$. So we have that f' has the same property as f. After a finite number of steps we will obtain the desired representation. □

Combining the previous two lemmas, we can write for any $i = 0, \ldots, t$:

$$f_i = g_i + \sum_{\deg(g_j) < \deg(f_i)} R_{j,i}\, g_j, \qquad \text{with } R_{j,i} \in k[y],\ \forall\ i, j.$$

So we can form a $(t+1) \times (t+1)$ transition matrix R, with entries in $k[y]$, such that:

$$(g_0, \ldots, g_t)R = (f_0, \ldots, f_t).$$

The matrix R is of the form:

	0	j_1	j_2	j_3 $\ldots$	j_p $\quad t$
0	$1 \ 0 \ \ldots$	$0 \ \ldots$	$0 \ \ldots$		$0 \ \ldots$
	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	$R_{1,j_1} \ \ldots$ $\vdots$ $R_{j_1-1,j_1} \ldots$	$R_{1,j_2} \ \ldots$ $\vdots$ $\ldots$	$\cdots\cdots$	$\ldots \ R_{1,t}$ $\vdots$ $\ldots \ R_{j_1-1,t}$
j_1	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	$\vdots$ $R_{j_2-1,j_2} \ldots$	$\cdots\cdots$	$\vdots$ $\ldots \ R_{j_2-1,t}$
j_2	0	0	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$	$\cdots\cdots$	$\ldots \ R_{j_3-1,t}$
j_3	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$ $\ldots \ R_{j_p-1,t}$
j_p … t	0	0	0	$\cdots\cdots$	$\begin{smallmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{smallmatrix}$

We indexed the rows and columns of the matrix starting from 0 for simplicity. Because $\text{in}(f_0) = x^t$ does not divide any of the monomials in the support of any f_i, with $i > 0$ we have that $R_{0,j} = 0,\ \forall\ j > 0$. The rest is just a consequence of the fact that I_0 is a lex-segment ideal. So we have:

$$\deg(f_0) = \ldots = \deg(f_{j_1-1}) < \deg(f_{j_1}) = \ldots < \deg(f_{j_2}) = \ldots .$$

The next step to prove injectivity is to prove that $R_{i,j} = 0$. Here is the plan for the next and final part in the proof of the injectivity:

Notice that the columns of the matrix $R(X + A)$ are syzygies for the g_i's. We subtract from these syzygies appropriate multiples of the columns of $X + B$ such that we obtain new syzygies of the g_i's, this time with entries in $k[y]$. So all the entries must be actually 0. These entries will be linear combinations of the $R_{i,j}$'s, with coefficients the entries of A and B and some y^{d_i}. Given the restrictions on the degrees of the entries in A and B we will deduce some limitations on the degrees of the $R_{i,j}$'s that could only take place if all of them are 0.

We will do this block by block, starting with the block formed by the columns $i_1, i_1 + 1, \dots, i_2 - 1$ of R, continuing with the next $i_3 - i_2$ columns of R and so on.

Let $i_1 \leq s \leq i_2 - 1$. When multiplying R with the sth column of $X + A$, we get entries of the form:

$$\alpha_{r,s} + \sum_{j=i_1}^{i_2-1} R_{r-1,j}\ \alpha_{j+1,s},$$

where $1 \leq r \leq t$. Notice that when $r = 1$ the entry is actually $\alpha_{1,s}$. Also for $r \geq i_2$ the entries will be $\alpha_{r,s}$, because $R_{r,j} = 0$ for $r \geq i_2$ and $j \leq i_2 - 1$. So these entries involve only the $R_{i,j}$'s with $i \in \{1, \dots, i_2 - 1\}$ and $j \in \{i_1, \dots, i_2 - 1\}$. That is why we can prove this block by block. Also some of the $R_{i,j}$'s in this block will be 0 because of the form of the matrix R. We will describe these cases later on.

As $\alpha_{s+1,s} = -x + \dots$, to cancel the x's in every entry we must subtract from this new syzygy: $R_{r-1,s}\times$ (the $(r - 1)$th column of $X + B$). We do this for every $r = 1, \dots, i_1$. So we obtain a new syzygy for the g_i's with the following entries:

$$\alpha_{r,s} + \sum_{j=i_1}^{i_2-1} R_{r-1,j}\ \alpha_{j+1,s} - \sum_{l=1}^{i_2-1} R_{l,s}\ \beta_{r,l}.$$

As the $-x$ appears first with coefficient $R_{r-1,s}$ and then in the second sum also with coefficient $R_{r-1,s}$, we can conclude that each entry is a polynomial in $k[y]$. As we just added and subtracted syzygies, we obtain again a syzygy. But as in the initial terms of the g_i's there appear different powers of x, we get that all entries must be 0. So we have the following equations:

$$\alpha_{r,s} + \sum_{j=i_1}^{i_2-1} R_{r-1,j}\ \alpha_{j+1,s} - \sum_{l=1}^{i_2-1} R_{l,s}\ \beta_{r,l} = 0.$$

Recall that in this part we will show that $R_{r,s} = 0$ for $r \neq s$ with $i_1 \leq s \leq i_2 - 1$ and $1 \leq r \leq i_2$.

The entries of the matrix R below the diagonal and some of the ones above are already 0. If we denote by:

$$j(r-1) := \mathrm{Min}\{j \in \mathcal{J} \mid j > r-1\},$$
$$\widetilde{j}(s) := \mathrm{Max}\{j \in \mathcal{J} \mid j \leq s\},$$

we can easily notice that the equations actually are:

$$a_{r,s} - b_{r,s} + \sum_{j=j(r-1)}^{i_2-1} R_{r-1,j}\ \alpha_{j+1,s} - \sum_{l=1}^{\widetilde{j}(s)-1} R_{l,s}\ \beta_{r,l} = 0.$$

We are considering any $A, B \in \mathscr{A}_{I_0}$, so some of the $\alpha_{i,j}$'s and $\beta_{i,j}$'s may be 0. But we will always have that $\deg(\alpha_{ii}) = \deg(\beta_{ii}) = d_i$. Notice also that if $s = i_1$, as $R_{r-1,s-1} \neq 0$ only if $r < s$, we get that $j(r-1) = i_1 = s$. This means that the first sum starts from $j = s$, thus α_{i_1,i_1} will not appear. So, in the first sum whenever $R_{r-1,s-1} \neq 0$, we have $\deg(\alpha_{s,s}) \leq 2$.

The degrees of the α's and β's are as follows:

- $\deg(\alpha_{s,s}) = d_s$.
- For $j = j(r-1), \ldots, i_2 - 1$, $j \neq s-1$ we have:

$$\deg(\alpha_{j+1,s}) \leq \begin{cases} \deg(\beta_{r,r}) & \text{if } j+1 \in J \quad \text{or} \\ \deg(\beta_{r,r}) - 1 & \text{if } j+1 \notin J. \end{cases}$$

- For $l = 1, \ldots, \widetilde{j}(s) - 1$, $l \neq r$ we have:

$$\deg(\beta_{r,l}) \leq \begin{cases} \deg(\beta_{r,r}) & \text{if } l \in J \text{ and } l < r \quad \text{or} \\ \deg(\beta_{r,r}) - 1 & \text{else.} \end{cases}$$

- $\deg(\beta_{r,r}) = d_r$.

Depending on r and s there are four types of equations. From the first three types we can deduce directly upper bounds on the degree of $R_{r,s}$. The fourth type may need to be modified in order to obtain such bounds.

Type 1: If ($s \notin \mathcal{J}$ and $r \in \mathcal{J}$), or ($r \in \mathcal{I}$) then, as ($d_r \geq 2$ and $d_s = 1$) or ($d_r \geq 3$ and $d_s \leq 2$) we get:

$$\deg(R_{r,s}) < \begin{cases} \deg(R_{r-1,j}) & \text{for some } j \in \{j(r-1), \ldots, i_2 - 1\}, \qquad \text{or} \\ \deg(R_{l,s}) & \text{for some } l \in \{1, \ldots, \widetilde{j}(s) - 1\}, l \neq r. \end{cases}$$

Type 2: If $s \notin \mathcal{J}$ and $r \notin \mathcal{J}$ then, as $d_r = 1$ and $d_s = 1$, we get:

$$\begin{cases} \deg(R_{r,s}) < \begin{cases} \deg(R_{r-1,j}) & \text{for some } j+1 \notin \mathcal{J}, \qquad\qquad \text{or} \\ \deg(R_{l,s}) & \text{for some } r \neq l \notin \mathcal{J},\ \text{or } l > r, \end{cases} \\ \text{or} \\ \deg(R_{r,s}) \leq \begin{cases} \deg(R_{r-1,j}) & \text{for some } j+1 \in \mathcal{J}, \qquad\qquad \text{or} \\ \deg(R_{l,s}) & \text{for some } l \in \mathcal{J},\ \text{and } l < r. \end{cases} \end{cases}$$

Type 3: If $s \in \mathcal{J}$ and $r \in \mathcal{J} \setminus \mathcal{I}$ then, as $d_r \geq 2$ and $d_s = 2$, we get:

$$\begin{cases} \deg(R_{r,s}) < \begin{cases} \deg(R_{r-1,j}) & \text{for some } j \neq s-1, \qquad \text{or} \\ \deg(R_{l,s}) & \text{for some } l \neq r, \end{cases} \\ \text{or} \\ \deg(R_{r,s}) \leq \deg(R_{r-1,s-1}). \end{cases}$$

Type 4: If $s \in \mathcal{J}$ and $r \notin \mathcal{J}$ then we have $d_r = 1$ and $d_s = 2$. So in this case we need to modify the original equation. As $\alpha_{s,s}$ is the coefficient of $R_{r-1,s-1}$ we can look at the equation where $R_{r-1,s-1}$ appears in the second sum, with coefficient $\beta_{r-1,r-1}$. There are two sub-cases:

> The equation for $R_{r-1,s-1}$ is an equation of type 1,2 or 3. That is $b_{r-1,r-1}$ has maximal degree among the coefficients of the $R_{i,j}$.
>
> The equation for $R_{r-1,s-1}$ is again an equation of type 4. In this case we can suppose inductively that we have already modified that equation such that $b_{r-1,r-1}$ has maximal degree among the coefficients of the $R_{i,j}$'s.

Now we look at $r-1$.

If $r-1 \in \mathcal{J}$, then we multiply our initial equation with $y^{d_{r-1}-2}$ and subtract from it the equation for $R_{r-1,s-1}$. In this case we will find that now $y^{d_{r-1}-2} b_{r,r}$ has maximal degree as we wanted.

If $r-1 \notin \mathcal{J}$, then we subtract from our equation the equation for $R_{r-1,s-1}$ multiplied by y. In this case, the new $R_{i,j}$'s that may have a coefficient of degree 2, are of the form $R_{i,j}$, with $i < r-1$. So by repeating this procedure we will reach at some point the previous case.

For this type of equations there are three kinds of conclusions that we can draw:

$$\begin{cases} \deg(R_{r,s}) < \text{ some } R_{i,j}, \\ \quad \text{or} \\ \deg(R_{r,s}) \leq \begin{cases} \deg(R_{i,j}) & \text{for some } j+1 \in \mathcal{J} \text{ and } i < r, \qquad \text{or} \\ \deg(R_{l,j}) & \text{for some } l \in \mathcal{J} \text{ and } j < s, \end{cases} \\ \quad \text{or} \\ \deg(R_{r,s}) = 0. \end{cases}$$

The third possibility comes from the fact that when $r-1 \in \mathcal{J}$, by multiplying the equation with $y^{d_{r-1}-2}$ and subtracting the equation for $R_{r-1,s-1}$, we get that the new coefficient of $R_{r,s}$, (i.e. $y^{d_{r-1}-2}b_{r,r}$) may have the same degree as the free term $(\alpha_{r-1,s-1} - \beta_{r-1,s-1})$. When $r-1 \notin \mathcal{J}$ we will find this situation by induction, when we arrive at an index $i \in \mathcal{J}$.

We were vague for the strict inequality, because we do not need to know the indices in that case.

Now we just have to see that these inequalities imply that $R_{r,s} = 0$. Because of the third possibility for the equations of type 4, we have to consider again two cases. Denote by $M := \max\{\deg(R_{i,j}) \mid 1 \leq r \leq i_2 \text{ and } i_1 \leq s \leq i_2 - 1\}$. The following remarks are the key to the last part of the proof.

Remark 3.4.

1. If r and s satisfy the conditions for the type 1 equations, then we have $\deg(R_{r,s}) < M$.
2. If $\deg(R_{r,s}) = M > 0$, then $\deg(R_{r,s}) = \deg(R_{i,j})$ for some $i < r$.

Case 1: $M = 0$. In this case we can use induction on r. We will not need to modify the equations of type 4.
If $r = 1$ then we are in the type 1 situation. So $\deg(R_{1,j}) < M = 0$.
Suppose $R_{i,j} = 0$ for all $i < r$. Then for all four types of equations, when we replace with 0 the $R_{i,j}$'s with $i < r$, we get equations of the form:

$$a_{r,s} - b_{r,s} - \sum_{l=r}^{\widetilde{j}(s)-1} R_{l,s}\ \beta_{r,l} = 0.$$

So, as by construction we have $\deg(\beta_{r,r}) > \deg(\beta_{l,r})$ if $l > r$ and also $\deg(\beta_{r,r}) > \deg(\beta_{r,s})$, we get again that if $R_{r,j} \neq 0$ then $\deg(R_{1,j}) < M = 0$. This means we have $R_{r,j} = 0$ for all $j \leq i_2 - 1$, $j \neq r$.

Case 2: $M > 0$. This means not all $R_{i,j}$'s are zero for $i \in \{1, \dots, i_2\}$ and $j \in \{i_1, \dots, i_2 - 1\}$. Choose $R_{r,s} \neq 0$ with $\deg(R_{r,s}) = M$. Then by the second remark we can find another $R_{r,s}$ with $\deg(R_{r',s}) = M$ and $r' < r$. We can repeat this until we find a $R_{i,j}$ with i and j as in the type 1 equations.

Thus, by the first remark we obtain a contradiction.

The proof proceeds in the same way with the next block, formed by columns indexed from i_2 to $i_3 - 1$, and so on until we prove that $R = id$.

This ends the long proof for the injectivity.

3.2.3 Proof of 3

Even if it would make things easier, we will no longer assume in the proof of the surjectivity that I_0 is a lex segment ideal.

We want to find for every ideal $I \subset k[x,y]$ such that $\text{in}(I) = I_0$, a Hilbert-Burch matrix of the form $X + A$ with $A \in \mathscr{A}_{I_0}$.

It is easy to see that we can find a Gröbner basis $\{f_0, \dots, f_t\}$ for I with $\text{in}(f_i) = x^{t-i}y^{m_i}$ and leading coefficient 1. Because of the form the leading terms of these polynomials, we can also suppose that the monomials in the support of the f_i's are not divisible by x^t (except for the leading term of f_0). Otherwise, if there exists an i such that $cx^{t+h}y^l \in \text{Supp}(f_i)$, for some $h, l \geq 0$ and $c \in k$, we modify f_i to be $f_i - cx^h y^l f_0$.

The S-polynomials $y^{d_i} f_{i-1} - x f_i$ have no term in their support divisible by x^{t+1}. So their reduction to 0 will be of the following form:

$$y^{d_i} f_{i-1} - x f_i + \sum_{j=0}^{t} a_{j,i} f_j = 0, \tag{3.2}$$

with $a_{i,j} \in k[y]$, $\forall\ i, j$ and $\text{in}(\beta_j f_j) \leq \text{in}(y^{d_i} f_{i-1} - x f_i)$, for all $j = 0, \dots, t$. The fact that $a_{i,j} \in k[y]$ follows by slightly changing the proof of Lemma 3.3.

These S-polynomials correspond to syzygies of the leading terms of the f_i's:

$$y^{d_i}(x^{t-i+1}y^{m_{i-1}}) - x(x^{t-i}y^{m_i}).$$

As these syzygies generate the syzygy module of $\text{in}(f_i)$, Schreyer's theorem implies that the equations (3.2) generate the syzygy module of the f_i's.

Setting these syzygies as columns of a matrix, we obtain a $(t+1) \times t$ matrix of the form $X + A$, where the entries of A are elements of $k[y]$. By the Hilbert-Burch theorem we have that the t-minors of this matrix generate the ideal I.

By the inequality of the leading terms in (3.2) we obtain the following restrictions on the degrees of the $a_{i,j}$:

$$\deg(a_{i,j}) \leq \begin{cases} i - j + m_j - m_{i-1} - 1 & \text{if } i \leq j, \\ i - j + m_j - m_{i-1} & \text{if } i > j. \end{cases} \tag{3.3}$$

Now we will show how to modify this matrix in order to obtain a new matrix $X + A'$ with $A' \in \mathscr{A}_{I_0}$. It is easy to see that elementary operations on the Hilbert-Burch matrix do not change the fact that the maximal minors generate the ideal.

To do this we will use a sequence of pairs of standard operations, that we will call reduction moves.

Take $i \neq j$, with $i \in \{1, \ldots, t+1\}$ and $j \in \{1, \ldots, t\}$. Suppose we have

$$\deg(a_{i,j}) \geq \begin{cases} d_i & \text{if } i < j, \\ d_j & \text{if } i > j. \end{cases} \tag{3.4}$$

If $i < j$, (resp. $i > j$) denote by $q_{i,j}$ the quotient of the division of $a_{i,j}$ by $y^{d_i} + a_{i,i}$, (resp. $y^{d_j} + a_{j,j}$). So we have:

$$a_{i,j} = \begin{cases} (y^{d_i} + a_{i,i})q_{i,j} + r_{i,j}, \text{ with } \deg(r_{i,j}) < d_i, & \text{if } i < j, \\ (y^{d_j} + a_{j,j})q_{i,j} + r_{i,j}, \text{ with } \deg(r_{i,j}) < d_j, & \text{if } i > j. \end{cases} \tag{3.5}$$

Notice that, as the degree of $a_{i,j}$ is bounded as in (3.3), we also have:

$$\deg(q_{i,j}) \leq \begin{cases} i - j + m_j - m_i - 1 & \text{if } i < j, \\ i - j + m_{j-1} - m_{i-1} & \text{if } i > j. \end{cases} \tag{3.6}$$

We will call a (i,j)-reduction move the following two standard operations:

If $i < j$:

- Add the ith column multiplied by $-q_{i,j}$ to the jth column.
- Add the $(j+1)$th row multiplied by $q_{i,j}$ to the $(i+1)$th row.

If $i > j$:

- Add the jth row multiplied by $-q_{i,j}$ to the ith row.
- If $j \geq 2$, add the $(i-1)$th column multiplied by $q_{i,j}$ to the $(j-1)$th column.

So in the second case, when $j = 1$ we only do the first move.

The first operation, reduces the degree of the entry in the position (i,j), by replacing $a_{i,j}$ with $r_{i,j}$. The second one cancels the multiple ox x that appeared in the position $(i+1,j)$ if $i < j$, (respectively the position $(i, j-1)$ if $i > j$) as a consequence of the first move.

Notice that, after each such reduction move, the degrees in the new matrix are still bounded as in (3.3). We take a look at what happens for $i < j$, the other case being similar.

For the first operation, for all $k = 1, \dots, t+1$, we have:

$$\deg(a_{k,i}q_{i,j}) \le (k-i+m_i-m_{k-1})+(i-j+m_j-m_i-1) = k-j+m_j-m_{k-1}-1.$$

For the second operation, for all $k = 1, \dots, t$, we have:

$$\deg(a_{j+1,k}q_{i,j}) \le j+1-k+m_k-m_j+i-j+m_j-m_i-1 = i+1-k+m_k-m_i-1.$$

As it is clear that every reduction move influences more elements, not just the one it is aimed at, we will have to determine which entries are influenced "most". This way, we will be able to conclude that after a finite sequence of reduction moves we will be able to reduce the degree of an entry by 1 and leave all other degrees as before. This way, in the end, we will be able to reduce the matrix to the desired form. Notice that, once the matrix is in $\mathscr{A}_{I_0}$, by definition we cannot make any more reduction moves.

Let us denote with $\mathrm{Red}_{i,j}$ the reduction moves. We will say that $\mathrm{Red}_{i,j}$ is maximal in (k,l) if $a_{k,l}$ is modified such that $\deg(a_{k,l})$ reaches the upper bound given in (3.3). It is easy to see that in order to get this, also $\deg(q_{i,j})$ has to reach the upper bound given in (3.6). Let us see where a reduction could be of maximal degree.

Case 1: $i < j$. We have to have $\deg(q_{i,j}) = i-j+m_j-m_i-1$ according to (3.6). By definition $\mathrm{Red}_{i,j}$ will act on the elements of the jth column and on those of the $(i+1)$th row. Let us first take a look at what happens on the jth column.

Let $k \in \{1, \dots, t+1\}$. We want to see what the degree of $a_{k,i}q_{i,j}$ could be:

If $k < i$

$$\begin{aligned}\deg(a_{k,i}q_{i,j}) &= k-i+m_i-m_{k-1}-1+i-j+m_j-m_i-1\\ &= (k-j+m_j-m_{k-1}-1)-1,\end{aligned}$$

so it cannot reach the upper bound in (3.3).

If $k \ge j$

$$\begin{aligned}\deg(a_{k,i}q_{i,j}) &= k-i+m_i-m_{k-1}+i-j+m_j-m_i-1\\ &= k-j+m_j-m_{k-1}-1,\end{aligned}$$

so it can be maximal only if $k < j$.

Let us now look at what happens on the $(i+1)$th row. Let $k \in \{1, \dots, t\}$. The degree of $a_{j+1,k}q_{i,j}$ could be:

If $k \le j+1$

$$\begin{aligned}\deg(a_{j+1,k}q_{i,j}) &= j+1-k+m_k-m_j+i-j+m_j-m_i-1\\ &= (i+1-k+m_k-m_i)-1,\end{aligned}$$

so it can reach the upper bound only if $k > i+1$.

If $k > j+1$

$$\begin{aligned} \deg(a_{j+1,k}q_{i,j}) &= j+1-k+m_k-m_j-1+i-j+m_j-m_i-1 \\ &= (i+1-k+m_k-m_i-1)-1, \end{aligned}$$

so it cannot be maximal.

So for the reduction moves that act above the diagonal the positions that could be maximal are:

$$\begin{array}{rl} (k,j) & \text{if } \ i<k<j \\ (i+1,k) & \text{if } \ i+1<k\leq j+1. \end{array}$$

Case 2: $i > j$. We have to have $\deg(q_{i,j}) = i-j+m_{j-1}-m_{i-1}$ according to (3.6). By definition $\mathrm{Red}_{i,j}$ will act on the elements of the ith row and on those of the $(j-1)$th column. Let us first take a look at what happens on the ith row.

Let $k \in \{1,\ldots,t\}$, $k \neq j$. We want to see what the degree of $a_{j,k}q_{i,j}$ could be:

If $k < j$

$$\begin{aligned} \deg(a_{j,k}q_{i,j}) &= j-k+m_k-m_{j-1}+i-j+m_{j-1}-m_{i-1} \\ &= i-k+m_k-m_{i-1}, \end{aligned}$$

so it reaches the upper bound in (3.3).

If $k > j$

$$\begin{aligned} \deg(a_{j,k}q_{i,j}) &= j-k+m_k-m_{j-1}-1+i-j+m_{j-1}-m_{i-1} \\ &= (i-k+m_k-m_{i-1})-1, \end{aligned}$$

so it can be maximal only if $k > i$.

Let us now look at what happens on the $(j-1)$th column. Let $k \in \{1,\ldots,t+1\}$. The degree of $a_{k,i-1}q_{i,j}$ could be:

If $k < i-1$

$$\begin{aligned} \deg(a_{k,i-1}q_{i,j}) &= k-i+1+m_{i-1}-m_{k-1}-1+i-j+m_{j-1}-m_{i-1} \\ &= k-j+1+m_{j-1}-m_{k-1}-1, \end{aligned}$$

so it can reach the upper bound only if $k < j-1$.

If $k \geq i-1$

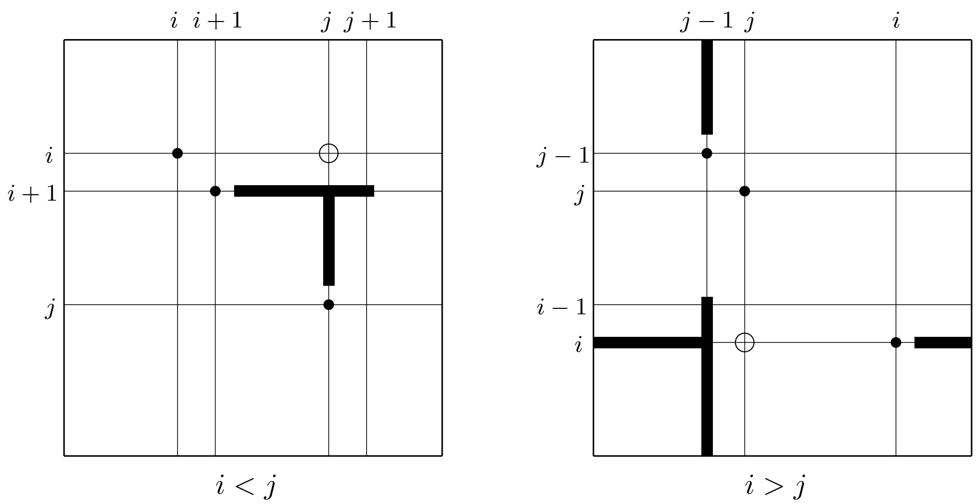
$$\begin{aligned} \deg(a_{k,i-1}q_{i,j}) &= k-i+1+m_{i-1}-m_{k-1}+i-j+m_{j-1}-m_{i-1} \\ &= k-j+1+m_{j-1}-m_{k-1}, \end{aligned}$$

so it can be maximal.

So for the reduction moves that act below the diagonal the positions that could be maximal are:

$$\begin{array}{rlll} (i,k) & \text{if } k<j & \text{or} & k>i \\ (k,j-1) & \text{if } k<j-1 & \text{or} & k\geq i-1. \end{array}$$

Here is a graphical representation of the positions that may be maximal for $\mathrm{Red}_{i,j}$:



The circle represents the position of the $a_{i,j}$ that is being reduced, the dots represent entries on the diagonal. The thin lines are columns, respectively rows, and the thick lines represent the positions in which maximal elements for $\mathrm{Red}_{i,j}$ may appear.

Now we will show how, using these reduction moves, we can bring the Hilbert-Burch matrix to the form we want to.

We will proceed by induction on t. When $t = 1$ there is not much to prove, so we can assume by induction that the upper left $t \times (t-1)$ part of the matrix is already in the form we want. We will show now how we can bring the elements of the last row and column to the desired form. We will start with the last row.

Suppose that for $j \in \{1, \ldots, t\}$ we have already brought the elements $a_{t+1,t}, \ldots, a_{t+1,j+1}$ to the desired degree. Also let us suppose that we have $\deg(a_{t+1,j}) = t + 1 - j + m_j - m_t > d_j - 1$.

First we do the reduction move $\mathrm{Red}_{t+1,j}$. This will have maximal degree. Then we will apply the other reduction moves that are necessary to bring the

$t\times(t-1)$ upper left part to the desired form. This can be done by induction. It is easy to see from the graphical representation, that for all these moves, the elements $a_{t+1,j},\dots,a_{t+1,t}$ will not be maximal. Now, also by induction we will bring to the desired form also the elements $a_{t+1,t},\dots,a_{t+1,j+1}$. Again, as the reduction moves will not be of maximal degree, by definition the element $a_{t+1,j}$ will not be maximal for any of them. So after performing all these reductions we will have $\deg(a_{t+1,j}) < t+1-i+m_j-m_t$.

This whole sequence of operations depends on the first reduction move $\text{Red}_{t+1,j}$. It is easy to notice that, even if we will start with a reduction that is not of maximal degree, we will still reduce the degree of $a_{t+1,j}$ by at least one. So we can do this until $\deg(a_{t+1,j}) \le d_j - 1$.

Now let us bring also the elements on the last column to the desired form. Suppose that the first $t-1$ columns and $a_{t+1,t}$ are of the desired form. Also suppose that we brought $a_{1,t},\dots,a_{i-1,t}$ to the desired form. Let $\deg(a_{i,t}) = i-t-1+m_t-m_{i-1} > d_i - 1$.

We apply now $\text{Red}_{i,t}$ which will be of maximal degree. Then we will bring the rest of the matrix, that we assumed had already the desired form, in the desired form again. These operations can be done by induction, and it is easy to see that the elements $a_{1,t},\dots,a_{i-1,t}$ will not be maximal. So also $a_{i,t}$ will not be maximal for any reduction. This means that we have reduced its degree by at least one.

This whole sequence of operations depends on the first reduction move $\text{Red}_{i,t}$, and it is easy to notice that, even if we will start with a reduction that is not of maximal degree, we will still reduce the degree of $a_{i,t}$ by at least one. So we can do this until $\deg(a_{i,t}) \le d_i - 1$. We have thus proven the surjectivity.

3.2.4 Dimension

In this part we will show how to compute the dimension of the affine space $V(I_0)$ that we parametrized. Using the same notation as in the proof of the main theorem we have:

Proposition 3.5. *Let $I_0 \subset R$ be a monomial lex-segment ideal. We have the following formula:*

$$\dim(V(I_0)) = \dim_k(R/I_0)+2(t+1)-\beta_{0,t+1}-2\beta_{0,t}+\sum_{i\ge 0}\left(\binom{\beta_{0,i}}{2}+\beta_{0,i+1}\beta_{0,i}\right)$$

where $\beta_{i,j} = \beta_{i,j}(I_0)$ are the graded Betti numbers of I_0. In particular $\beta_{0,i}$ is the number of minimal generators of I_0 that have degree i.

Proof. To compute the dimension of $V(I_0)$ we just have to look at the number of coefficients that appear in a matrix of $\mathscr{A}_{I_0}$. For this, we will use the same notations that we used in the proof of 2. We recall that

$$\begin{aligned}
\mathcal{I} &= \{i_0, \ldots, i_q\} := \{i \in 1, \ldots, t \mid d_i \geq 3\} \cup \{1\},\\
\mathcal{J} &= \{j \in 1, \ldots, t \mid d_j \geq 2\},\\
j(l) &= \mathrm{Min}\{j \in \mathcal{J} \mid j > l\},\\
\widetilde{j}(l) &= \mathrm{Max}\{j \in \mathcal{J} \mid j \leq l\}.
\end{aligned}$$

This means that we have:

$$\begin{array}{rcl}
\deg(x^t) & = \ldots = & \deg(x^{t-j_1+1}y^{m_{j_1-1}}) < \\
< \deg(x^{t-j_1}y^{m_{j_1}}) & = \ldots = & \deg(x^{t-j_2+1}y^{m_{j_2-1}}) < \\
< \deg(x^{t-j_2}y^{m_{j_2}}) & = \ldots = & \deg(x^{t-j_3+1}y^{m_{j_3-1}}) < \\
& \ldots & \\
< \deg(x^{t-j_q}y^{m_{j_q}}) & = \ldots = & \deg(y^{m_t}),
\end{array}$$

where $\mathcal{J} = \{j_1, j_2, \ldots, j_q\}$. Denote by $\mathcal{J}^* = \{j_0, j_1, \ldots, j_q, j_{q+1}\}$ where $j_0 = 1$, $j_{q+1} = t+1$ and the others are just the elements of $\mathcal{J}$. As I_0 is a lex-segment ideal, for $i \in \{1, \ldots, q\}$:

$$\begin{aligned}
j_{i+1} - j_i &= \text{number of minimal generators of degree } \; t - j_i + m_{j_i}\\
&= \beta_{0,t-j_i+m_{j_i}}(I_0).
\end{aligned}$$

We also have:

$$\begin{aligned}
j_1 - j_0 + 1 &= \text{number of minimal generators of degree } \; t\\
&= \beta_{0,t}(I_0).
\end{aligned}$$

Let us now look at the form of a matrix in $\mathscr{A}_{I_0}$. We have:

$$\deg(a_{k,l}) \leq \begin{cases}
d_k - 1 & \text{if } k \leq l,\\
1 & \text{if } l = j_s \in \mathcal{J} \text{ and } j_s + 1 \leq k \leq j_{s+1},\\
0 & \text{if } l \notin \mathcal{J} \text{ and } l + 1 \leq k \leq j(l),\\
0 & \text{if } j(l) = j_s \in \mathcal{J} \setminus \mathcal{I} \text{ and } j_s + 1 \leq k \leq j_{s+1},\\
-1 & \text{otherwise.}
\end{cases}$$

The first case are the entries on and above the diagonal. As a polynomial in $k[y]$ of degree at most r has $r+1$ coefficients, form the first case we get:

$$\sum_{i=1}^{t} (t - i + 1) d_i = \sum_{i=1}^{t} m_i = \dim_k(R/I_0).$$

From the second case, which corresponds to entries below $y^{d_j} + a_{j,j}$, with $j \in \mathcal{J}$, that may have degree 1, we get:

$$\sum_{i=2}^{q+1} 2(j_i - j_{i-1}) = \sum_{i \neq t} 2\beta_{0,i} = 2\sum \beta_{0,i} - 2\beta_{0,t} = 2(t+1) - 2\beta_{0,i}.$$

The third case, which corresponds to entries below $y + a_{l,l}$ with $l \notin \mathcal{J}$, gives us:

$$\sum_{i=1}^{q+1} \frac{(j_i - j_{i-1})(j_i - j_{i-1} - 1)}{2} = \sum_{i \geq 0} \binom{\beta_{0,i}}{2}.$$

And finally, the last part will count the constants in the rectangular blocks below the diagonal. These blocks appear only if there are generators in two consecutive degrees. We have:

$$\begin{aligned} \sum_{i=1}^{q} (j_i - j_{i-1})(j_{i+1} - j_i) &= \beta_{0,t+1}(\beta_{0,t} - 1) + \sum_{i \neq t} \beta_{0,i}\beta_{0,i+1} \\ &= \sum_{i \geq 0} \beta_{0,i}\beta_{0,i+1} - \beta_{0,t+1}. \end{aligned}$$

To conclude, we only have to add up the results in the four different cases. □

3.2.5 Examples

We will show now with three examples how the proof of the main theorem works. We start with a "small" example from which it will be easier to see the main idea behind the proof of the injectivity. Then, we are forced to choose a rather "large" example in order to present the more technical arguments that we use in the proof. The last example shows how to find the canonical Hilbert-Burch matrix for a given ideal I, i.e. the corresponding matrix $A \in \mathscr{A}_{\text{in}(I)}$.

Example 1

Let I_0 be the following ideal:

$$I_0 = (x^3, x^2y^5, xy^7, y^{11}).$$

So we have: $m_0 = 0$, $m_1 = 5$, $m_2 = 7$, $m_3 = 11$ and $d_1 = 5$, $d_2 = 2$, $d_3 = 4$. The sets of "special" indices are: $\mathcal{I} = \{1, 3\}$ and $\mathcal{J} = \{1, 2, 3\}$. Let us compute the dimension in this case. We have $\dim_k(R/I_0) = 5 + 7 + 11 = 23$. The non-zero Betti numbers are: $\beta_{0,3} = \beta_{0,7} = \beta_{0,8} = \beta_{0,11} = 1$. So by applying the formula we get:

$$\dim(V(I_0)) = 23 + 2 \cdot 4 - 0 - 2 \cdot 1 + 1 = 30.$$

The matrix that bounds the degrees of the entries of a matrix $A \in \mathscr{A}_{I_0}$ is:

$$\begin{pmatrix} 4 & 4 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \\ -3 & -2 & 1 \end{pmatrix}.$$

By looking at this matrix we can check that the formula of the dimension is correct in this case.
Now let A and B be two matrices in $\mathscr{A}_{I_0}$. The matrix $X + A$ will be:

$$X + A = \begin{pmatrix} y^5 + a_{1,1} & a_{1,2} & a_{1,3} \\ -x + a_{2,1} & y^2 + a_{2,2} & a_{2,3} \\ a_{3,1} & -x + a_{3,2} & y^4 + a_{3,3} \\ 0 & 0 & -x + a_{4,3} \end{pmatrix}.$$

The matrix $X + B$ will have a similar form. Using the same notations as in the proof we can write:

$$\begin{array}{rcl} f_0 & = & g_0, \\ f_1 & = & g_1, \\ f_2 & = & g_2 + R_{1,2}\ g_1, \\ f_3 & = & g_3 + R_{1,3}\ g_1 + R_{2,3}\ g_2. \end{array}$$

So the transition matrix R will have the following form:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & R_{1,2} & R_{1,3} \\ 0 & 0 & 1 & R_{2,3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that, as the columns of $X+A$ are syzygies for (f_0, f_1, f_2, f_3), the columns of $R(X + A)$ will be syzygies for (g_0, g_1, g_2, g_3). From these we will subtract the necessary multiples of the columns of B in order to obtain syzygies with entries in $k[y]$:

$$S_1 = \begin{pmatrix} y^5 + a_{1,1} \\ -x + a_{2,1} + a_{3,1}R_{1,2} \\ a_{3,1} \\ 0 \end{pmatrix} - \begin{pmatrix} y^5 + b_{1,1} \\ -x + b_{2,1} \\ b_{3,1} \\ 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} a_{1,2} \\ y^2 + a_{2,2} + (-x + a_{3,2})R_{1,2} \\ -x + a_{3,2} \\ 0 \end{pmatrix} - \begin{pmatrix} b_{1,2} \\ y^2 + b_{2,2} \\ -x + b_{3,2} \\ 0 \end{pmatrix} - \begin{pmatrix} (y^5 + b_{1,1})R_{1,2} \\ (-x + b_{2,1})R_{1,2} \\ b_{3,1}\ R_{1,2} \\ 0 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} a_{1,3} \\ a_{2,3} + (y^4 + a_{3,3})R_{1,2} + (-x + a_{4,3})R_{1,3} \\ y^4 + a_{3,3} + (-x + a_{4,3})R_{2,3} \\ -x + a_{4,3} \end{pmatrix} - \begin{pmatrix} (y^5 + b_{1,1})R_{1,3} \\ (-x + b_{2,1})R_{1,3} \\ b_{3,1}\ R_{1,3} \\ 0 \end{pmatrix} -$$

$$- \begin{pmatrix} b_{1,2}\ R_{2,3} \\ (y^2 + b_{2,2})R_{2,3} \\ (-x + b_{3,2})R_{2,3} \\ 0 \end{pmatrix} - \begin{pmatrix} b_{1,3} \\ b_{2,3} \\ y^4 + b_{3,3} \\ -x + b_{4,3} \end{pmatrix}.$$

From S_1 we get that:

$$a_{2,1} + a_{3,1}R_{1,2} - b_{1,2} = 0.$$

But we cannot draw any conclusion from here, as $a_{3,1}$ may also be 0.

From the first entry of S_2 we have

$$a_{1,2} - b_{1,2} - (y^5 + b_{1,1})R_{1,2} = 0.$$

As $\deg(a_{1,2}) \leq 4$ and $\deg(b_{1,2}) \leq 4$ we obtain that $R_{1,2} = 0$. We set $R_{1,2} = 0$ in S_3 and we get

$$\begin{aligned} a_{1,3} - b_{1,3} - (y^5 + b_{1,1})R_{1,3} - b_{1,2}R_{2,3} &= 0, \\ a_{2,3} + a_{4,3}R_{1,3} - b_{2,1}R_{1,3} - (y^2 + b_{2,2})R_{2,3} - b_{2,3} &= 0. \end{aligned}$$

From the first equation, as all the a's and b's have degree less then 4 we get that if $R_{1,3} \neq 0$ then

$$\deg(R_{1,3}) < \deg(R_{2,3}).$$

From the second equation, as this time all the a's and b's have degree less then 1 we get that if $R_{2,3} \neq 0$ then

$$\deg(R_{2,3}) < \deg(R_{1,3}).$$

This means that we actually must have $R_{1,3} = R_{2,3} = 0$.

Example 2

In the previous example, we did not have to change the equations. Also, as all indices were "special" in that case, we obtained directly strict inequalities. In the next example we will see all the possible types of situations that may arise. Let I_0 be the following ideal:

$$\begin{aligned} I_0 &= (x^{12}, x^{11}y^3, x^{10}y^4, x^9y^5, x^8y^{10}, x^7y^{11}, x^6y^{12}, \\ &\quad x^5y^{14}, x^4y^{15}, x^3y^{16}, x^2y^{19}, xy^{20}, y^{21}). \end{aligned}$$

So $t = 12$, the m's and the d's are:

$$\begin{aligned} m &= (0, 3, 4, 5, 10, 11, 12, 14, 15, 16, 19, 20, 21), \\ d &= (3, 1, 1, 5, 1, 1, 2, 1, 1, 3, 1, 1). \end{aligned}$$

The sets of "special" indices are:

$$\begin{aligned} \mathcal{I} &= \{1, 4, 10\}, \\ \mathcal{J} &= \{1, 4, 7, 10\}. \end{aligned}$$

Let us again compute the dimension of $V(I_0)$. In this case $\dim_k(R/I_0) = 150$. The non-zero Betti numbers are: $\beta_{0,12} = 1$ and $\beta_{0,14} = \beta_{0,18} = \beta_{0,19} = \beta_{0,21} = 3$. So the dimension will be:

$$\dim(V(I_0)) = 150 + 2 \cdot 13 - 0 - 2 \cdot 1 + 3 + (3 + 9) + 3 + 3 = 195.$$

One can again verify this is true by looking at the matrix that bounds the degrees of the matrices in $\mathscr{A}_{I_0}$. We will now look at a general matrix $A \in \mathscr{A}_{I_0}$. This will be an 13×12 matrix, with entries polynomials in $k[y]$. The matrix that bounds the degrees of the $a_{i,j}$'s is the following:

$$\begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
-3 & -3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & -3 & -3 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4 & -4 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-4 & -4 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-4 & -4 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\
-6 & -6 & -6 & -2 & -2 & -2 & -1 & -1 & -1 & 1 & 0 & 0 \\
-6 & -6 & -6 & -2 & -2 & -2 & -1 & -1 & -1 & 1 & 0 & 0 \\
-6 & -6 & -6 & -2 & -2 & -2 & -1 & -1 & -1 & 1 & 0 & 0
\end{pmatrix}.$$

In order to emphasize the maximal possible degree of each $a_{i,j} \neq 0$ we denote:

$$a_{i,j} = \begin{cases} \bullet\, y^{m_{i,j}} & \text{, if } m_{i,j} > 0, \\ c & \text{, if } m_{i,j} = 0. \end{cases}$$

The dot stands for the coefficient of the highest power of y. With this notation the matrix A is:

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
\bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 \\
\hline
\bullet\, y & c & c & c & c & c & c & c & c & c & c & c \\
\bullet\, y & c & c & c & c & c & c & c & c & c & c & c \\
\bullet\, y & c & c & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 & \bullet\, y^4 \\
\hline
0 & 0 & 0 & \bullet\, y & c & c & c & c & c & c & c & c \\
0 & 0 & 0 & \bullet\, y & c & c & c & c & c & c & c & c \\
0 & 0 & 0 & \bullet\, y & c & c & \bullet\, y & \bullet\, y & \bullet\, y & \bullet\, y & \bullet\, y & \bullet\, y \\
\hline
0 & 0 & 0 & c & c & c & \bullet\, y & c & c & c & c & c \\
0 & 0 & 0 & c & c & c & \bullet\, y & c & c & c & c & c \\
0 & 0 & 0 & c & c & c & \bullet\, y & c & c & \bullet\, y^2 & \bullet\, y^2 & \bullet\, y^2 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet\, y & c & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet\, y & c & c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet\, y & c & c
\end{array}\right).$$

Let $B \in \mathscr{A}_{I_0}$ be another matrix as in the proof of the injectivity. Suppose that they both parametrize the same ideal. The transition matrix R from $X+A$ to $X+B$ is:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & R_{1,4} & R_{1,5} & R_{1,6} & R_{1,7} & R_{1,8} & R_{1,9} & R_{1,10} & R_{1,11} & R_{1,12} \\
0 & 0 & 1 & 0 & R_{2,4} & R_{2,5} & R_{2,6} & R_{2,7} & R_{2,8} & R_{2,9} & R_{2,10} & R_{2,11} & R_{2,12} \\
0 & 0 & 0 & 1 & R_{3,4} & R_{3,5} & R_{3,6} & R_{3,7} & R_{3,8} & R_{3,9} & R_{3,10} & R_{3,11} & R_{3,12} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & R_{4,7} & R_{4,8} & R_{4,9} & R_{4,10} & R_{4,11} & R_{4,12} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & R_{5,7} & R_{5,8} & R_{5,9} & R_{5,10} & R_{5,11} & R_{5,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & R_{6,7} & R_{6,8} & R_{6,9} & R_{6,10} & R_{6,11} & R_{6,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & R_{7,10} & R_{7,11} & R_{7,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & R_{8,10} & R_{8,11} & R_{8,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & R_{9,10} & R_{9,11} & R_{9,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Now we take a look at the syzygies that arise from multiplying the transition matrix with the matrix $X+A$. It is clear that when multiplying with the first three columns of $X+A$, the R's do not appear. Also when multiplying columns 4 to 9, the last three columns of the matrix R do not play any role. So in this example we will look just at the $R_{i,j}$'s with $j \leq 9$. We will now follow the steps of the proof of the injectivity.

When me multiply the 4th column of $X+A$ with the transition matrix we obtain:

$$\begin{array}{l}
\bullet\, y^2 \\
c + (-x + \bullet\, y)R_{1,4} + \bullet\, yR_{1,5} + \bullet\, yR_{1,6} + cR_{1,7} + cR_{1,8} + cR_{1,9} \\
c + (-x + \bullet\, y)R_{2,4} + \bullet\, yR_{2,5} + \bullet\, yR_{2,6} + cR_{2,7} + cR_{2,8} + cR_{2,9} \\
y^5 + (-x + \bullet\, y)R_{3,4} + \bullet\, yR_{3,5} + \bullet\, yR_{3,6} + cR_{3,7} + cR_{3,8} + cR_{3,9} \\
(-x + \bullet\, y) + cR_{4,7} + cR_{4,8} + cR_{4,9} \\
\bullet\, y + cR_{5,7} + cR_{5,8} + cR_{5,9} \\
\bullet\, y + cR_{6,7} + cR_{6,8} + cR_{6,9} \\
c \\
c \\
c \\
0 \\
0 \\
0
\end{array}$$

To cancel the x's that appear, we add to this syzygy:

$$\begin{aligned}(-R_{1,4}) \text{ times the first column of } & X+B,\\ (-R_{2,4}) \text{ times the 2nd column of } & X+B,\\ (-R_{3,4}) \text{ times the 3rd column of } & X+B,\\ \text{4th column of } & X+B.\end{aligned}$$

As this column vector will be a syzygy with entries polynomials in y, we obtain that all the entries must be 0. So we obtain the following equations:

$$\bullet\, y^2 - \mathbf{y^3 R_{1,4}} - \bullet\, y^2 R_{2,4} - \bullet\, y^2 R_{3,4} = 0,$$

$$\begin{aligned}c + \bullet\, yR_{1,4} + \bullet\, yR_{1,5} + \bullet\, yR_{1,6} + cR_{1,7} + cR_{1,8} + cR_{1,9}&\\ -\bullet\, yR_{1,4} - \mathbf{yR_{2,4}} - cR_{3,4} &= 0,\end{aligned}$$

$$\begin{aligned}c + \bullet\, yR_{2,4} + \bullet\, yR_{2,5} + \bullet\, yR_{2,6} + cR_{2,7} + cR_{2,8} + cR_{2,9}&\\ -\bullet\, yR_{1,4} - cR_{2,4} - \mathbf{yR_{3,4}} &= 0.\end{aligned}$$

We obtain also other equations, but these will be the only equations that we actually can deduce something from. Notice that the summands in bold do not depend on the choice of the entries of A or B. The coefficients of $R_{1,4}, R_{2,4}$ and $R_{3,4}$ are of the highest degree. This means that:

If $R_{1,4} \neq 0$ then

$$\deg(R_{1,4}) < \begin{cases} \deg(R_{2,4}), & \text{or} \\ \deg(R_{3,4}). \end{cases}$$

If $R_{2,4} \neq 0$ then

$$\begin{cases} \deg(R_{2,4}) < \begin{cases} \deg(R_{1,j}) & \text{for some } j \in \{7,8,9\}, \\ \deg(R_{3,4}), \end{cases} & \text{or} \\ \quad \text{or} & \\ \deg(R_{2,4}) \leq \deg(R_{1,j}) \text{ for some } j \in \{4,5,6\}. & \end{cases}$$

If $R_{3,4} \neq 0$ then

$$\begin{cases} \deg(R_{3,4}) < \deg(R_{2,j}) \text{ for some } j \in \{7,8,9\}, & \\ \quad \text{or} & \\ \deg(R_{3,4}) \leq \begin{cases} \deg(R_{2,j}) & \text{for some } j \in \{4,5,6\}, \\ \deg(R_{1,4}). \end{cases} & \text{or} \end{cases}$$

Most of the relations will be obtained in the same way. The only different situation will appear when we multiply the matrix R with the 7th column of $X + A$. In this case we will obtain two equations which we need to modify:

$$\begin{aligned} c + cR_{1,4} + cR_{1,5} + \underline{y^2 R_{1,6}} + \bullet\, yR_{1,7} + \bullet\, yR_{1,8} + \bullet\, yR_{1,9} - & \\ - \mathbf{yR_{2,7}} - cR_{3,7} - cR_{4,7} - cR_{5,7} - cR_{6,7} &= 0, \quad (3.7) \\ c + cR_{2,4} + cR_{2,5} + \underline{y^2 R_{2,6}} + \bullet\, yR_{2,7} + \bullet\, yR_{2,8} + \bullet\, yR_{2,9} - & \\ - \bullet\, yR_{1,7} - \mathbf{yR_{3,7}} - cR_{4,7} - cR_{5,7} - cR_{6,7} &= 0. \quad (3.8) \end{aligned}$$

By the proof, we want to get inequalities on the degrees of $R_{2,7}$, respectively $R_{3,7}$ from the equations (3.7) and (3.8). But the degree of their coefficients is not maximal among the other coefficients. To correct this we will use the following equations:

$$\begin{aligned} \bullet\, y^2 - \mathbf{y^3 R_{1,5}} - \bullet\, y^2 R_{2,5} - \bullet\, y^2 R_{3,5} &= 0, \quad (3.9) \\ \bullet\, y^2 - \mathbf{y^3 R_{1,6}} - \bullet\, y^2 R_{2,6} - \bullet\, y^2 R_{3,6} &= 0, \quad (3.10) \\ +cR_{1,4} + \bullet\, yR_{1,5} + \bullet\, yR_{1,6} + cR_{1,7} + cR_{1,8} + cR_{1,9} - & \\ - \mathbf{yR_{2,6}} - cR_{3,6} &= 0. \quad (3.11) \end{aligned}$$

To obtain a "good" equation for $R_{2,7}$ we multiply (3.7) by y and subtract (3.10). To obtain a "good" for $R_{3,7}$ we multiply (3.11) by y and subtract it from (3.8). Then, as me may still have the coefficients of $R_{1,5}$ and $R_{1,6}$ of higher degree then the one of $R_{3,7}$, we multiply the new equation by y and subtract from it (3.9) times a constant and (3.10) times a constant. And this will be enough, as in both (3.9) and (3.10) the coefficients of $R_{1,5}$, respectively $R_{1,6}$ are of degree strictly higher then the other coefficients that may appear. This way the equation of $R_{2,7}$ will become:

$$\begin{aligned} & \bullet\, y^2 + \bullet\, yR_{1,4} + \bullet\, yR_{1,5} + \bullet\, y^2 R_{1,6} + \bullet\, y^2 R_{1,7} + \bullet\, y^2 R_{1,8} + \\ & \bullet\, y^2 R_{1,9} - \mathbf{y^2 R_{2,7}} - \bullet\, yR_{3,7} - \bullet\, yR_{4,7} - \bullet\, yR_{5,7} - \bullet\, yR_{6,7} - \\ & \qquad - \bullet\, y^2 R_{2,6} - \bullet\, y^2 R_{3,6} = 0. \end{aligned} \quad (3.12)$$

One can see that now we obtain either inequalities on the degree of $R_{2,7}$ or that $R_{2,7}$ is of degree 0. This is why we have to consider two cases.

Denote by M the maximum of the degrees of the R's. The first case is when $M = 0$. In this case we obtain that all the R's are 0 without changing the original equations. First we get that $R_{1,4} = \ldots = R_{1,9} = 0$ and $R_{4,7} = \ldots = R_{4,9} = 0$ because their degree cannot be maximal. Then, by looking at what is left of the equations of the $R_{2,j}$'s we get that also $R_{2,4} = \ldots = R_{2,9} = 0$. And so on.

If $M > 0$ then we look at the R's with maximal degree. So the fact that $R_{2,7}$ could be of degree 0 (i.e. not maximal), does not change the proof.

Suppose for instance that $R_{3,8}$ is of maximal degree. So its degree cannot be strictly less then the degree of another $R_{i,j}$. So we have:

$$\deg(R_{3,8}) = \begin{cases} \deg(R_{1,8}), & \text{or} \\ \deg(R_{2,6}), & \text{or} \\ \deg(R_{2,7}). \end{cases}$$

The degree of $R_{1,8}$ cannot be maximal.
If $R_{2,7}$ is a constant, we cannot have this equality either. So $\deg(R_{2,6}) = M$. But then

$$\deg(R_{2,6}) = \begin{cases} \deg(R_{1,6}), & \text{or} \\ \deg(R_{1,5}). \end{cases}$$

And none of those can be maximal.
If $R_{2,7}$ would be of maximal degree we get by (3.12) that

$$\deg(R_{2,7}) = \begin{cases} \deg(R_{1,j}) & \text{for some } j \in \{6, \dots, 9\}, \qquad \text{or} \\ \deg(R_{2,6}), & \text{or} \\ \deg(R_{3,6}). \end{cases}$$

Again it cannot be any of the $R_{1,j}$'s, and as we have seen neither $R_{2,6}$ that are of maximal degree. So it must be $R_{3,6}$. But then we have:

$$\deg(R_{3,6}) = \begin{cases} \deg(R_{1,6}), & \text{or} \\ \deg(R_{2,5}). \end{cases}$$

So by the same argument we must have $\deg(R_{2,5}) = M$. But then we get that $\deg(R_{1,4}) = M$ or $\deg(R_{1,5}) = M$ - a contradiction.

It is easy to notice that if some of the $R_{i,j}$ would be 0, this would only reduce the number of cases we have to consider.

Example 3

Now we will give an example of how the proof of the surjectivity of ψ works. That is we will start with an ideal $I \subset R$ with $\dim(R/\operatorname{in}(I)) = 0$ and construct the corresponding matrix of $\mathscr{A}_{\operatorname{in}(I)}$.

Let I be the ideal generated by the following polynomials:

$$\begin{aligned} f_0 &= x^3 - x^2y - 2xy^2 + 2y^3 - 2x^2 + xy + y^2 - x + 2y - 2, \\ f_1 &= x^2y^2 - 2y^4 - x^3 + x^2y - 2y^3 + x^2 - 3xy + 4y^2 + 4x - y, \\ f_2 &= xy^3 - y^4 - 2x^2y + 6xy^2 - 5y^3 + x^2 - xy + 2y^2 - 3x + 4y - 2, \\ f_3 &= y^5 + x^2y^2 - 2xy^3 + 2y^4 + 3xy^2 + 2y^3 - x^2 - 2xy - y^2 - x - 11y + 6. \end{aligned}$$

These polynomials are already a DRL Gröbner basis for I. So its initial ideal will be $I_0 = \operatorname{in}(I) = (x^3, x^2y^2, xy^3, y^5)$.We have: $t = 3$, $m_0 = 0$,

$m_1=2$, $m_2=3$, $m_3=5$ and $d_1=2$, $d_2=1$, $d_3=2$. Notice first that in the support of f_1 there is a monomial divisible by a power of x higher than or equal to t: x^3. So we will set f_1 to be f_1+f_0.

The next step is to compute the S-polynomials:

$$\begin{aligned} S_{1,0} &= y^2 f_0 - x f_1, \\ S_{2,1} &= y\ \ f_1 - x f_2, \\ S_{3,2} &= y^2 f_2 - x f_3. \end{aligned}$$

After performing the division algorithm we obtain:

$$\begin{array}{rcrcrcrcr} S_{1,0} &=& (-1)f_0 &+& yf_1 &+& f_2 &+& 0f_3, \\ S_{2,1} &=& (-2y+1)f_0 &+& f_1 &+& (-y+1)f_2 &+& f_3, \\ S_{3,2} &=& (y^2-1)f_0 &+& 3f_1 &+& f_2 &+& (y+1)f_3. \end{array}$$

By Schreyer's theorem, these syzygies generate the syzygy module of I. So we have obtained the following Hilbert-Burch matrix:

$$\begin{pmatrix} y^2-1 & -2y+1 & y^2-1 \\ -x+y & y+1 & 3 \\ 1 & -x-y+1 & y^2+1 \\ 0 & 1 & -x+y+1 \end{pmatrix}.$$

Notice that, as expected, it is a matrix of the form $X+A$. The matrix that bounds the degrees of the entries of the matrices in $\mathscr{A}_{I_0}$ is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} -1 & -2y+1 & y^2-1 \\ +y & 1 & 3 \\ 1 & -y+1 & 1 \\ 0 & 1 & -y+1 \end{pmatrix},$$

so $A \notin \mathscr{A}_{I_0}$. We will need to do some reduction moves. We will start looking at the upper left 2×1 corner of A. There the bounds are respected. Now we will look at the up upper left 3×2 corner. We start looking at the last row of this block, from right to left. Then, if everything is fine there, we look at the last column from top to bottom. In this case, the first entry that we look at, $(a_{3,2})$ has degree higher than the bound. So we apply the reduction move $\mathrm{Red}_{3,2}$ to $X+A$:

- Subtract from row 3, row 2 multiplied by (-1) .
- As you can see, in position 3,1 there is an entry which contains x. So to cancel this x we subtract from column 1: column 2 multiplied by (1). We obtain:

$$\begin{pmatrix} y^2-1 & -2y+1 & y^2-1 \\ -x+y & y+1 & 3 \\ -x+y+1 & -x+2 & y^2+4 \\ 0 & 1 & -x+y+1 \end{pmatrix}, \text{ then } \begin{pmatrix} y^2+2y-2 & -2y+1 & y^2-1 \\ -x-1 & y+1 & 3 \\ y-1 & -x+2 & y^2+4 \\ -1 & 1 & -x+y+1 \end{pmatrix}.$$

Now we start over with checking the matrix. This time we find an entry with degree higher than the bound in position 1,3. We apply $\mathrm{Red}_{1,3}$:

- Subtract from column 3, column 1 multiplied by 1.

- Subtract from row 2, row 4 multiplied by 1. We obtain:

$$\begin{pmatrix} y^2+2y-2 & -2y+1 & -2y+1 \\ -x-1 & y+1 & x+4 \\ y-1 & -x+2 & y^2-y+5 \\ -1 & 1 & -x+y+2 \end{pmatrix}, \text{ then } \begin{pmatrix} y^2+2y-2 & -2y+1 & -2y+1 \\ -x-2 & y+2 & y+6 \\ y-1 & -x+2 & y^2-y+5 \\ -1 & 1 & -x+y+2 \end{pmatrix}.$$

We check again the matrix in the same order and find that the entry 2,3 does not respect the upper bound. Notice that this entry was of lower degree when we started. So we apply now $\mathrm{Red}_{2,3}$:

- Subtract from column 3, column 2 multiplied by (-1).

- Subtract from row 3, row 4 multiplied by (-1). We obtain:

$$\begin{pmatrix} y^2+2y-2 & -2y+1 & 0 \\ -x-2 & y+2 & 4 \\ y-1 & -x+2 & y^2+x-y+3 \\ -1 & 1 & -x+y+1 \end{pmatrix}, \text{ then } \begin{pmatrix} y^2+2y-2 & -2y+1 & 0 \\ -x-2 & y+2 & 4 \\ y-2 & -x+3 & y^2+4 \\ -1 & 1 & -x+y+1 \end{pmatrix}.$$

And now, after checking again, we find that this time the matrix respects all the upper bounds. So the matrix $A' \in \mathscr{A}_{I_0}$ that corresponds to the ideal I is:

$$\begin{pmatrix} 2y-2 & -2y+1 & 0 \\ -2 & 2 & 4 \\ y-2 & 3 & 4 \\ -1 & 1 & y+1 \end{pmatrix}.$$

The generators of I given by the signed minors of the Hilbert-Burch matrix have changed. They are now:

$$\begin{aligned}
f_0' &= x^3 - x^2y - 2xy^2 + 2y^3 - 2x^2 + xy + y^2 - x + 2y - 2, \\
f_1' &= x^2y^2 - xy^3 - y^4 + 2x^2y - 8xy^2 + 5y^3 - 2x^2 - xy + 3y^2 + 6x - 3y, \\
f_2' &= xy^3 - y^4 - 2x^2y + 6xy^2 - 5y^3 + x^2 - xy + 2y^2 - 3x + 4y - 2, \\
f_3' &= y^5 - 2xy^3 + 4y^4 + 5xy^2 + 2y^3 - 6y^2 - 4x - 12y + 8.
\end{aligned}$$

3.3 Ideals in $k[x, y, z]$

In this section we will consider ideals of the polynomial ring in three variables. Given any monomial ideal J_0 of $k[x, y, z]$ and considering the affine variety of the homogeneous ideals that have I_0 as initial ideal for a certain term order τ, we do not obtain in general the affine space (see [8] and [17] for examples). We will prove that if we take $J_0 = I_0k[x, y, z]$, with $I_0 \in k[x, y]$

a lex-segment ideal, and choose the degree reverse lexicographic order induced by $x > y > z$, then $V_h(J_0)$ is again the affine space. We also give a parametrization for this space, which comes from the parametrization of $V(I_0)$.

First we will introduce some notation and recall some results that we will use.

3.3.1 Notation and useful results

We will denote by $S := k[x, y, z]$ and, as before, $R = k[x, y]$. We present now some known results on homogenization and dehomogenization. Most of them can be found in a more general form in [32].

Definition 3.6. Let $f \in R$ and $F \in S$ be two polynomials. We will write $f = c_1t_1 + \ldots + c_st_s$, with $c_i \in k$ and t_i monomials in x and y. Denote $u_i := \deg(t_i)$ and set $\mu := \max\{u_i\}$.

a) We define the homogenization of f in S to be:

$$f^{hom} := \sum_{i=1}^{s} c_i t_i z^{\mu - u_i}.$$

b) We define de dehomogenization of F with respect to the variable z to be:

$$F^{deh} := F(x, y, 1).$$

Here are some remarks on the behavior of polynomials under the two operations defined above.

Proposition 3.7. *Consider $f, g \in R$ and $F, G \in S$.*

1. $(f^{hom})^{deh} = f$.
2. *If* $s = \max\{i \mid z^i \text{ divides } F\}$ *then:* $z^s(F^{deh})^{hom} = F$.
3. $(fg)^{hom} = f^{hom}g^{hom}$.
4. $(FG)^{deh} = F^{deh}G^{deh}$.
5. $(F + G)^{deh} = F^{deh} + G^{deh}$.

Now we will extend these two operations to ideals.

Definition 3.8. Let $I \subset R$ and $J \subset S$ be two ideals.

a) We define the homogenization of I in S to be the ideal:

$$I^{hom} := (f^{hom} \mid f \in I) \subseteq S).$$

b) We define de dehomogenization of J with respect to the variable z to be the set:
$$J^{deh} := \{F^{deh} \mid F \in J\} \subseteq R\}.$$

Notice that J^{deh}, being the image of J under a surjective ring homomorphism from S to R, is also an ideal. We have the following proposition:

Proposition 3.9. *Let $I \subset R$ and $J \subset S$ be two ideals.*

1. $(I^{hom})^{deh} = I$.
2. $J \subseteq (J^{deh})^{hom} = J :_S (z)^{\infty}$.
3. *If $I \neq R$ then z is a non-zero divisor of S/I^{hom}.*

On both R and S we will consider from now on the degree reverse lexicographic term order. As this term order is degree compatible, from [32], Chapter 4.3 we can deduce the following.

Lemma 3.10. *Let $f \in R$ a non-zero polynomial and $F \in S$ a non-zero homogeneous polynomial. Then*

1. $\mathrm{in}(f^{hom}) = \mathrm{in}(f)$.
2. $\mathrm{in}(F^{deh}) = (\mathrm{in}(F))^{deh}$.

The last two results that we will cite in this section will regard the homogenization and dehomogenization of Gröbner basis.

Proposition 3.11. *Let I be an ideal of R and let J be a non-zero homogeneous ideal of S .*

1. *If $\{f_1, \dots, f_s\}$ is a Gröbner basis of I, then $\{f_1^{hom}, \dots, f_s^{hom}\}$ is a Gröbner basis of I^{hom}.*
2. *If $\{F_1, \dots, F_s\}$ is a homogeneous Gröbner basis of J, then $\{F_1^{deh}, \dots, F_s^{deh}\}$ is a Gröbner basis of J^{deh}.*

Now we will define similar operations on matrices. In particular, the dehomogenization of a matrix A with entries S with respect to the variable z will be just the dehomogenization of all its entries. We will denote this new matrix, with entries in R by A^{deh}.

The homogenization of a matrix with entries in R will not be defined this straight forward. We will define this only for the matrices that parametrize $V(I_0)$.

Let $I_0 \subseteq R$ be a monomial ideal generated, as in the previous chapter, by $x^t, x^{t-1}y^{m_1}, \dots, y^{m_t}$. We define its degree matrix to be the $(t+1) \times t$ matrix U with entries:
$$u_{i,j} = m_j - m_{i-1} + i - j.$$
Now we can define the homogenization of a matrix. Notice that this will depend on the degree matrix associated to I_0.

Definition 3.12. Let $A \in \mathscr{A}_{I_0}$, with entries $a_{i,j}$. For every $i = 1, \dots, t+1$ and $j = 1, \dots, t$ we define:

$$a_{i,j}^{\underline{hom}} := z^{u_{i,j} - \deg(a_{i,j})} a_{i,j}^{hom},$$

where $a_{i,j}^{hom}$ is the standard homogenization defined in 3.6. The homogenization of the matrix A will be the matrix with entries $a_{i,j}^{\underline{hom}}$. We will denote this matrix by $A^{\underline{hom}}$.

Remark 3.13. We could define the homogenization in the same way also for the matrix $X + A$. But as the entries of X are either 0 or of degree $u_{i,j}$ we would have:

$$(X + A)^{\underline{hom}} = X + (A^{\underline{hom}}).$$

The matrices $A^{\underline{hom}}$ and $X + A^{\underline{hom}}$ are homogeneous matrices in the sense of Definition 4.7.1. of [32]. So their minors will be homogeneous polynomials in S. In particular, the ideal generated by the maximal minors of $X + A^{\underline{hom}}$ is a homogeneous ideal of S.

For $i = 0, \dots, t$ will denote by f_i the determinant of the matrix obtained from $X + A$ by deleting the $(i+1)$th row times $(-1)^{i+1}$, and by F_i the determinant of the matrix obtained from $X + A^{\underline{hom}}$ by deleting the $(i+1)$th row times $(-1)^{i+1}$. It is easy to see that we have:

$$F_i = (f_i)^{hom}.$$

We will end this section with a lemma that will turn out useful later.

Lemma 3.14. *Let $A \in \mathscr{A}_{I_0}$ be a matrix. With the above notations we have:*

1. $(I_t(X + A))^{hom} = I_t(X + A^{\underline{hom}})$.
2. $I_t(X + A) = (I_t(X + A^{\underline{hom}}))^{deh}$.

Proof. In the proof of Theorem 3.1 we have seen that the set $\{f_0, \dots, f_t\}$ forms a degree reverse lexicographic Gröbner basis of $I_t(X + A)$. So, by Proposition 3.11 we have that the set $\{F_0, \dots, F_t\}$ forms a degree reverse lexicographic Gröbner basis of $(I_t(X + A))^{hom}$. Thus the first part follows. The second part is an immediate consequence of the first point of Proposition 3.9. □

3.3.2 Parametrization

Using the parametrization given by Theorem 3.1, we will now parametrize the following variety. Let $I \subset S$ be an ideal "of points" in $\mathbb{P}^2$ such that z is not a zero divisor for S/I. That is I is a Cohen-Macaulay homogeneous ideal and $\dim(S/I) = 1$.

Remark 3.15. The fact that z is not a zero divisor for S/I is equivalent to $\mathrm{in}(I)$ is generated by monomials that are not divisible by z.

Proof. If there would exist a minimal generator of $\mathrm{in}(I)$ divisible by z, given the fact that we use the degree reverse lexicographic term order, we would find a homogeneous generator of I that would be a multiple of z.
Now suppose z is a zero divisor, and choose $f \in S \setminus I$ such that $zf \in I$ and $\mathrm{in}(f)$ is minimal with this property. As $z\,\mathrm{in}(f) \in \mathrm{in}(I)$, which is generated by monomials in x and y, we obtain $\mathrm{in}(f) \in \mathrm{in}(I)$. So there exists a polynomial $g \in I$ with $\mathrm{in}(f) = \mathrm{in}(g)$. As $f-g \notin I$, $z(f-g) \in I$ and $\mathrm{in}(f-g) < \mathrm{in}(f)$ we obtain a contradiction. □

Denote $\mathrm{in}(I) = J_0$. The ideal J_0 will be of the form:

$$J_0 = I_0 S, \quad \text{with } I_0 \subset R, \text{ a monomial ideal.}$$

We will consider the ideals for which I_0 is just as in the hypothesis of Theorem 3.1. So we have that $\dim(R/I_0) = 0$ and we will also require I_0 to be a lex-segment ideal. For this type of ideals we will parametrize the following affine variety:

$$\overline{V}(J_0) = \{I \subset S \mid I \text{ is a homogeneous ideal with } \mathrm{in}(I) = I_0 S\}.$$

We will prove that this variety is parametrized also by $\mathscr{A}_{I_0}$. Recall that $\mathscr{A}_{I_0}$ was the set of matrices with entries polynomials in y, that satisfy (3.1). We define the following application:

$$\overline{\psi} : \mathscr{A}_{I_0} \longrightarrow \overline{V}(J_0)$$
$$\overline{\psi}(A) = I_t(X + A^{\underline{hom}}), \quad \text{for all } A \in \mathscr{A}_{I_0}.$$

Theorem 3.16. *The application $\overline{\psi} : \mathscr{A}_{I_0} \longrightarrow \overline{V}(J_0)$ defined above is a bijection.*

Proof. In order to prove the theorem we need to prove again three things:

1. The application $\overline{\psi}$ is well defined.

2. The application $\overline{\psi}$ is injective.

3. The application $\overline{\psi}$ is surjective.

Proof of 1. For every A in $\mathscr{A}_{I_0}$, denote the ideal $I_t(X + A^{\underline{hom}})$ by I_A. We need to show that I_A is homogeneous and has $\mathrm{in}(I_A) = J_0$. Using the notations in the previous section we have by definition that the polynomials $F_0, \ldots, F_t$ are homogeneous. We just need to show that they form a Gröbner basis and that their initial terms generate J_0.

We know from Theorem 3.1 that $f_0, \ldots, f_t$ form a Gröbner basis of I_0. As we have seen that for $i = 0, \ldots, t$ we have $(f_i)^{hom} = F_i$, by applying Proposition 3.11 we get that $F_0, \ldots, F_t$ form a Gröbner basis. And by applying Lemma 3.10 we obtain:

$$\text{in}(f_i) = \text{in}(F_i), \quad \text{for all} \ \ i = 0, \ldots, t.$$

Proof of 2. Let A and B be two matrices in $\mathscr{A}_{I_0}$. Suppose that $\overline{\psi}(A) = \overline{\psi}(B)$. That is we have:

$$I_t(X + A^{\underline{hom}}) = I_t(X + B^{\underline{hom}}).$$

By the second part of Lemma 3.14 we obtain that we also have:

$$I_t(X + A) = I_t(X + B.)$$

And by the injectivity of ψ we get that $A = B$.

Proof of 3. Let $I \in \overline{V}(J_0)$ be a homogeneous ideal. By Proposition 3.11 we have that $I^{deh} \subset R$ is an ideal that has $\text{in}(I^{deh}) = I_0$. So by Theorem 3.1 we know that

$$I^{deh} = I_t(X + A), \quad \text{for some} \ \ A \in \mathscr{A}_{I_0}.$$

We will show that $I = I_t(X + A^{\underline{hom}})$. By Lemma 3.14 we have that $I_t(X + A^{\underline{hom}}) = (I_t(X + A))^{hom} = (I^{deh})^{hom}$. To complete the proof we just need to show that $I = (I^{deh})^{hom}$. By Proposition 3.9 this means we have to show that $I = I :_S (z)^\infty$. But this is equivalent to z not being a zero divisor for S/I. □

3.4 Betti strata

We will now fix a Hilbert series H and consider all the ideals I of points in $\mathbb{P}^2$ such that the Hilbert series of S/I is H. By an ideal of points in $\mathbb{P}^2$ we understand a homogeneous ideal $I \subset S$ such that S/I is Cohen-Macaulay of dimension 1. In this case we have that the maximal ideal $\underline{m} = (x_1, \ldots, x_n)$ of S is not an associated prime of S/I. So, the Hilbert series H will be of the form:

$$H(s) = \frac{h(s)}{1 - s},$$

with $h(t)$ the Hilbert series of the 0-dimensional algebra $S/I + (\ell)$, where ℓ is a linear non-zero divisor of S/I. Denote by

$$\mathbb{G}(H) = \{I \subset S \mid I \text{ is an ideal of points with } H_{S/I} = H\}$$

the variety that parametrizes graded homogeneous ideals of S such that the Hilbert series of S/I is H.

The first restriction that we will use will be to consider ideals for which z is not a zero divisor. Let I be an ideal of points in $\mathbb{P}^2$. This also means that

$$I = q_1 \cap \ldots \cap q_s,$$

where for all i we have $\sqrt{q_i} = p_i$ and p_i is the ideal of a point P_i in $\mathbb{P}^2$. The geometric equivalent of z not being a zero divisor for S/I is that none of the points $P_1, \ldots, P_s$ belongs to the line of $\mathbb{P}^2$ given by $z = 0$. This means that the set:

$$\mathbb{G}^*(H) := \{I \in \mathbb{G}(H) \mid z \text{ is a non-zero divisor for } S/I\}$$

is an open subset of $\mathbb{G}(H)$.

The fact that z is not a zero divisor for S/I implies that $\text{in}(I) = JS$, where J is an ideal of R. We also have that $H_{R/J}(s) = h(s)$. The same thing also holds for the degree reverse lexicographic generic initial ideal of I. So we have that:

$$\text{Gin}(I) = J_0 S, \quad \text{where } J_0 \subset R.$$

But in characteristic 0 the generic initial ideal is strongly stable, so we also get that J_0 must be strongly stable. As $R = k[x, y]$, the only strongly stable ideal with that Hilbert series is $\text{Lex}(h)$. This means that the set:

$$\mathbb{G}^*_{Lex}(H) = \{I \in \mathbb{G}^*(H) \mid \text{in}(I) = \text{Lex}(h)S\}$$

is an open subset of $\mathbb{G}^*(H)$.

We will study now the Betti strata of this affine set. For a homogeneous ideal $I \subset S$ we will denote by $\beta_{i,j}(I)$ the (i, j)th Betti number. In particular, $\beta_{0,j}(I)$ is the number of minimal generators of I of degree j. It is known that any two of the sets $\{\beta_{0,j}(I)\}_j$, $\{\beta_{1,j}(I)\}_j$ and $\{\dim(I_j)\}_j$ determine the third. For the fixed Hilbert series $H(s) = h(s)/(1-s)$ and for given integers j and u we define:

$$\begin{aligned} V(H, j, u) &= \{I \in \mathbb{G}^*_{Lex}(H) \mid \beta_{0,j}(I) = u\}, \\ V(H, j, \geq u) &= \{I \in \mathbb{G}^*_{Lex}(H) \mid \beta_{0,j}(I) \geq u\}. \end{aligned}$$

For a vector $\beta = (\beta_1, \ldots, \beta_j, \ldots)$ with integral entries we define:

$$\begin{aligned} V(H, \beta) &= \textstyle\bigcap_j \quad V(H, j, \beta_j), \\ V(H, \geq \beta) &= \textstyle\bigcap_j \quad V(H, j, \geq \beta_j). \end{aligned}$$

For the fixed Hilbert function H, we consider the lex-segment ideal $\text{Lex}(h)$ and denote by $m_0, \ldots, m_t$ its associated sequence. We have shown that $\mathbb{G}^*_{Lex}(H)$ is parametrized by $\mathscr{A}_{\text{Lex}(H)}$, which is an affine space $\mathbb{A}^n$. We will denote the coordinates of $\mathbb{A}^n$ by $a_1, \dots, a_n$. So we know that to each ideal $I \in \mathbb{G}^*_{Lex}(H)$ corresponds a unique matrix $A \in \mathscr{A}_{\mathrm{Lex}(H)}$. Starting from this matrix A we can construct a Hilbert-Burch matrix, that is $X + A^{\underline{hom}}$. For simplicity we will denote:

$$M := X + A^{\underline{hom}}.$$

We have by the Hilbert-Burch theorem the following free resolution:

$$0 \longrightarrow \bigoplus_{i=1}^{t} S(-q_i) \xrightarrow{M} \bigoplus_{i=1}^{t+1} S(-p_i) \longrightarrow I \longrightarrow 0 \tag{3.13}$$

where $p_i = t+1-i+m_i$ for $i = 1, \dots, t+1$ and $q_i = p_i + 1$ for $i = 1, \dots, t$. For every integer j we set:

$$w_j = \{i \mid p_i = j\} \quad \text{and} \quad v_j = \{i \mid q_i = j\}.$$

For every integer j denote by M_j the submatrix of M with row indices w_j and column indices v_j. As we are considering matrices that are in $\mathscr{A}_{\mathrm{Lex}(h)}$ we know that $0 = m_0 < m_1 < \dots < m_t$. So we also get $t+1 = p_0 \leq p_1 \leq \dots \leq p_t$. This means that we can describe the matrices M_j in terms of the m_i's. Even more, the entries of these matrices will be independent coordinates of $\mathbb{A}^n$.

To compute the graded Betti numbers of I we can tensor the resolution (3.13) with k and look at the degree j component. This will give us the following complex of vector spaces, whose homology gives the Betti numbers of I:

$$k^{\sharp v_j} \xrightarrow{M_j} k^{\sharp w_j} \longrightarrow 0.$$

Hence we have that:

$$\beta_{0,j}(I) = \sharp w_j - \mathrm{rank}(M_j).$$

This means that $\beta_{0,j}(I) \geq u$ is equivalent to

$$\mathrm{rank}(M_j) \leq \sharp w_j - u.$$

Notice that, as we start from a matrix $A \in \mathscr{A}_{Lex(h)}$, we have $\sharp w_j = \beta_{0,j}(\mathrm{Lex}(h))$. That is the number of minimal generators of degree j of $\mathrm{Lex}(h)$. We also have $\sharp v_j = \beta_{1,j}(\mathrm{Lex}(h)) = \beta_{0,j-1}(\mathrm{Lex}(h))$.

So we obtain that $V(H, j, \geq u)$ is the determinantal variety given by the following condition on the $\beta_{0,j}(\mathrm{Lex}(h)) \times \beta_{0,j-1}(\mathrm{Lex}(h))$ matrix:

$$\mathrm{rank}(M_j) \leq \beta_{0,j}(\mathrm{Lex}(h)) - u.$$

It is easy to notice, that for $i \neq j$ the sets of variables involved in M_i and M_j are disjoint. This means that the intersection $\bigcap_j V(H, j, \geq \beta_j)$ is transversal. We have proven the following:

Proposition 3.17. *Each $V(H, j, \geq \beta_j)$ is a determinantal variety and the variety $V(H, \geq \beta)$ is the transversal intersection of the $V(H, j, \geq \beta_j)$'s. The variety $V(H, j, \geq \beta_j)$ is irreducible and it coincides with the closure of $V(H, j, \beta_j)$, provided $V(H, j, \beta_j)$ is not empty.*

Corollary 3.18. *The variety $V(H, \geq \beta)$ is irreducible.*

Appendix A

A CoCoA Program for Chapter 3

In this appendix we present the implementation of an algorithm which computes the canonical Hilbert-Burch of Theorem 3.1. The algorithm is written using the language of the computer algebra program CoCoA.

We start with an ideal $I \subset R$ with $\dim_k(R/I) < \infty$, and check whether its initial ideal is a lex-segment ideal. The command `StampaBene` is just a way to print matrices in a clear way.

```
Define StampaBene(X,P);
Foreach A In X Do
 PrintLn ;
 Foreach B In A Do
  Print Format(B,P);
End; End End;

Use R::=Q[x,y];
ID:=Ideal( .. );

Ts:=0;
Repeat Ts:=Ts+1; Until x^Ts IsIn LT(ID);
If Len(Gens(ID)) = Ts+1 Then
     PrintLn("The initial ideal is a Lex-segment ideal");
     Else PrintLn("Find another example. You started wrong.");
End;
```

In the next step, we compute a Gröbner basis of the ideal. We index the polynomials in the Gröbner basis such that the power of x in the leading term of f_i is greater than the power of x in the leading term of f_j when $i < j$. Then, using the same notation as in Chapter 3, we compute t, the m_i's and the d_i's.

```
L:=GBasis(ID);
F:=NewList(Len(L));
For I:=1 To Len(F) Do
    E:=Log(LT(L[I]));
    F[Len(F) - E[1]]:= L[I]/LC(L[I]);
EndFor;

T:=Len(F)-1;
M:=NewList(T+1);
For I:=1 To T+1 Do  E:=Log(LT(F[I]));  M[I]:=E[2]; End;
D:=NewList(T);
For I:=1 To T Do  D[I]:=M[I+1]-M[I]; EndFor;
```

Now we modify the Gröbner basis $\{f_0, \ldots, f_t\}$, such that no monomial, except $\mathrm{in}(f_0)$, in the support of any f_i is divisible by a higher power of x than $t-1$.

```
FW:=F;
COUNT:=0;

For I := 2 To T+1 Do
 Repeat
  K:=0;
  Test:=0;
  FW[I]:=F[I];
  While FW[I] <> 0 And Test = 0 Do
   FW[I]:=FW[I] - LM(FW[I]);
   If FW[I] <> 0 Then
    E:=Log(LT(FW[I]));
    H:=E[1] -T;
    L:=E[2];
   EndIf;
   If H >= 0 And FW[I] <> 0  Then
    K:=K+1;
    COUNT:=COUNT+1;
    Test:=Test+1;
    F[I]:=F[I] - LC(FW[I])x^H y^L  F[1];
   EndIf;
  EndWhile;
 Until K = 0;
EndFor;

PrintLn("We had To change the generators ",COUNT," Times");
```

Now we compute a Hilbert-Burch matrix. In most cases this will not be the canonical Hilbert-Burch matrix that we are looking for. What we do is compute the generators of the syzygy module by dividing the S-polynomials $S_{i,i+1}$ for $i = 0, \ldots, t-1$ by the Gröbner basis. Then, just to be sure, we check that the signed minors of this matrix are the polynomials we started with.

```
S:=NewList(Len(F)-1);
For J:=1 To Len(S) Do
    S[J]:= - (LT(F[J+1])/GCD(LT(F[J]),LT(F[J+1])))F[J] +
             (LT(F[J])/GCD(LT(F[J]),LT(F[J+1])))F[J+1];
End;

B:=NewList(Len(F)-1);
For I:= 1 To Len(S) Do
    DivAlg(S[I],F);
    DIV:=It;
    B[I]:=DIV.Quotients;
EndFor;
B:=Transposed(Mat(B));

For I:= 1 To T Do
        B[I,I]:=y^D[I] + B[I,I]; B[I+1,I]:=-x+B[I+1,I];
End;

TF:=Reversed(Minors(T,B));
For I:=1 To T+1 Do
  If IsEven(T-I) Then
    TF[I]:=-TF[I];
  End;
End;

TF = F;
```

Now we bring this matrix to the canonical form, in the way described in the proof of 3. We let `B` unchanged, in order to compare the results in the end, and operate on the matrix `A`. The matrix `GA` is the matrix that gives the degree limits for the matrices in $\mathscr{A}_{I_0}$. The counter `NB` tells us how many times we had to operate on the matrix. In the end we will also check that the new matrix defines the same ideal that we started with.

```
A:=B;

GA:=Mat[[ -1 | I In 1..T ] | J In 1..(T+1) ];
```

```
For I:=1 To T+1 Do
 For J:=1 To T Do
  If I<=J Then
   GA[I,J]:=Min(I-J-1+M[J+1]-M[I], D[I]-1);
   Else GA[I,J]:=Min(I-J+M[J+1]-M[I], D[J]-1);
  End;
 End;
End;

K:=0;
NB:=0;

Repeat
Test:=0;
K:=0;
 For I:= 2 To T+1 Do
  If Test = 0 Then
   For J:= 1 To I-1 Do
    If Test = 0 Then
     K:=K+1;
     If  A[I,I-J] <> 0 And Deg(A[I,I-J],y) > GA[I,I-J] Then
      Test:=Test+1;
      NB:=NB+1;
      C:=(A[I,I-J]-NR(A[I,I-J],[A[I-J,I-J]]))/A[I-J,I-J];
      For N:= 1 To T Do
       A[I,N]:= A[I,N] - C*A[I-J,N];
      EndFor;
      PrintLn('_________________');
      PrintLn('Q = ', C);
      PrintLn('I = ', I);
      PrintLn('J = ',I-J);
      StampaBene(A,25);
      PrintLn;
      If I-J > 1 Then
       For N:= 1 To T+1 Do
        A[N,I-J-1]:=A[N,I-J-1] + C*A[N,I-1];
       EndFor;
      PrintLn('_________________');
      PrintLn('Q = ', C);
      StampaBene(A,25);
      PrintLn;
      EndIf;
     EndIf;
```

```
    EndIf;
   EndFor;

   If I>2 Then
    For J:= 1 To I-2 Do
     If Test = 0 Then
      K:=K+1;
      If A[J,I-1] <> 0 And Deg(A[J,I-1],y) > GA[J,I-1] Then
       NB:=NB+1;
       Test:=Test+1;
       C:=(A[J,I-1]-NR(A[J,I-1],[A[J,J]]))/A[J,J];
       For N:= 1 To T+1 Do
        A[N,I-1]:=A[N,I-1] - C*A[N,J];
       EndFor;
       PrintLn('_________________');
       PrintLn('Q = ', C);
       PrintLn('I = ', J);
       PrintLn('J = ',I-1);
       StampaBene(A,25);
       PrintLn;
       For N:= 1 To T Do
        A[J+1,N]:=A[J+1,N]+ C*A[I,N];
       EndFor;
       PrintLn('_________________');
       PrintLn('Q = ', C);
       StampaBene(A,25);
       PrintLn;
      EndIf;
     EndIf;
    EndFor;
   EndIf;
  EndIf;
 EndFor;
Until K=T^2;

FID:= Ideal(Minors(T,A));
FID=ID;

PrintLn("We had To operate On the matrix ", NB ,"  Times");
```

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