Freie Universität Berlin
Institut für Mathematik
Stand: 15. Januar 2025

WiSe24/25
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Algebra I – Homework 12

Deadline: **20:00 on Wednesday 22.01.2025.** (Uploads are still possible until Friday 24.01 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H12.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let R be a Noetherian ring and let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of R. Consider all possible chains of primary ideals

$$\mathfrak{q} = \mathfrak{q}_n \subsetneq \mathfrak{q}_{n-1} \subsetneq \cdots \subsetneq \mathfrak{q}_0 = \mathfrak{p}$$

1. Show that there exists a global bound $r \in \mathbb{N}$ such every such chain has length $n \leq r$.
2. Show that all maximal chains as above have the same length.

Problem 2. **2 points**

Show that an ideal I in a Noetherian ring R is primary if and only if R/I has exactly one associated prime.

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Describe a composition series of $\mathbb{Z}/n\mathbb{Z}$ in terms of the prime-factor decomposition of $n \in \mathbb{N}_{>1}$.

Extra Problem 4.

Let R be a Noetherian ring and let $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Prove that f is nilpotent if and only if each a_i is nilpotent.

Extra Problem 5.

What are the minimal and what are the associated primes $\mathfrak{p}$ of $R = \mathbb{C}[x,y]/(x^2, xy^2)$? For all of the latter provide an embedding $R/\mathfrak{p} \hookrightarrow R$. Which of the localizations $R_\mathfrak{p}$ have finite length – and what is this length? Visualize a monomial $\mathbb{C}$-base of R and all $R/\mathfrak{p}$ – how does this reflect the previous information about the lengths?

Extra Problem 6.

Let M, N be finitely generated modules over a Noetherian ring R. Show that this implies that $\operatorname{Ass}\operatorname{Hom}_R(M,N) = (\operatorname{Supp} M) \cap (\operatorname{Ass} N)$.

(**Hint:** Using the compatibility of Ass with localizations, we may assume that $R = (R, \mathfrak{m})$ is local, and that it is sufficient to check under which conditions $\mathfrak{m}$ is contained in either side of the equation.)

Extra Problem 7.

1. If $R[x]$ is Noetherian, does this imply that R is Noetherian?

2. If $R_\mathfrak{p}$ is Noetherian for all $\mathfrak{p} \in \operatorname{Spec} R$, does this imply that R is Noetherian?