Freie Universität Berlin
Institut für Mathematik
Stand: 8. Januar 2025

WiSe24/25
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Algebra I – Homework 11

Deadline: **20:00 on Wednesday 15.01.2025.** (Uploads are still possible until Friday 17.01 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H11.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Let M be an R-module and $f : M \longrightarrow M$ be an R-module homomorphism. Show that:

1. If M is Noetherian and f is surjective, then f is an isomoprhism.
2. If M is Artinian and f is injective, then f is an isomoprhism.

Problem 2. **2 points**

Let $\mathbb{K}$ be a field, and let $R = \mathbb{K} \oplus x^2 \cdot \mathbb{K}[x]$ and $S = \mathbb{K} \oplus xy \cdot \mathbb{K}[x, y]$ be two $\mathbb{K}$-algebras.

1. Check if R or S are finitely generated $\mathbb{K}$-algebras.
2. Check if R and S are Noetherian rings.

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Show that for any poset $(\mathcal{P}, \leqslant)$ the ascending chain condition is equivalent to the maximal condition and that the descending chain condition is equivalent to the minimal condition.

Extra Problem 4.

Let M be an R-module. Show that, if every non-empty set of finitely generated submodules of M has a maximal element, then M is Noetherian.

Extra Problem 5.

1. Show that any ring R has a Noetherian R-module.
2. Show that a ring R is Noetherian if and only if there exists a faithful (i.e. $\operatorname{Ann}_R M = 0$) Noetherian R-module M.

Extra Problem 6.

A *topological space* X is **Noetherian** if the open subsets of X satisfy the ascending chain condition. Show that if R is a Noetherian ring, then $\operatorname{Spec}(A)$ (with the Zariski topology) is a Noetherian topological space. Is the converse also true?

Extra Problem 7.

Show that a domain R is Artinian if and only if it is a field.