Freie Universität Berlin
Institut für Mathematik
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WiSe24/25
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Algebra I – Homework 10

Deadline: **20:00 on Wednesday 8.01.2025.** (Uploads are still possible until Friday 10.01 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H10.pdf`

Full written proofs are required in support of your answers.

Problem 1. — 2 points

Let $\mathbb{K}$ be a field, $I = (x^2, xy) \subseteq \mathbb{K}[x,y]$, and $R = \mathbb{K}[x,y]/I$.

1. Show that (y^n) is $(x,y)-$primary[‡] in R, and that $(0) = (x) \cap (y^n)$ is a minimal primary decomposition for every $n \geqslant 1$.
2. Show that $(x + \lambda y^n)$ is also $(x,y)-$primary for any nonzero $\lambda \in \mathbb{K}$, and that $(0) = (x) \cap (x + \lambda y^n)$ in R.

Problem 2. — 2 points

Show that if a decomposable ideal is radical, then it has no embedded primes.

Total: 4 points

[‡] All ideals and polynomials from $\mathbb{K}[x,y]$ are intended as their image under $\mathbb{K}[x,y] \longrightarrow R$.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let X be an infinite compact Hausdorff space and $\mathcal{C}(X)$ be the ring of real-valued functions on X. Is the zero ideal decomposable in $\mathcal{C}(X)$?

Extra Problem 4.

Let $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, and $\mathfrak{m} = (x, y, z)$ be ideals of $\mathbb{K}[x, y, z]$. Let $I = \mathfrak{p}_1 \cdot \mathfrak{p}_2$.

1. Show that $\mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$ is a minimal primary decomposition of I.
2. Show that $\bigcup_{i\geqslant 1}(I : \mathfrak{m}^i)$ is an ideal and compute a monomial generating set and a minimal primary decomposition.

Extra Problem 5.

One can extend the notion of associated primes to R-modules in the following way:

*A prime $\mathfrak{p} \in \operatorname{Spec} R$ is **associated** to the R-module M, if $\mathfrak{p}$ is the annihilator of an element of M.*

We will denote the set of associated primes of the R-module M by

$$\operatorname{Ass}_R M = \{\mathfrak{p} \in \operatorname{Spec} R \ : \ \exists x \in M \text{ such that } \mathfrak{p} = \operatorname{Ann}_R x\}.$$

Note that this does not coincide with the definition of associated prime of an ideal, if we view the ideal as an R-module. This exception is widely and traditionally used, and the risk of confusion is low. What we actually have is:

$$\operatorname{Ass} I = \operatorname{Ass}_R R/I,$$

where on the left hand side we use the definition of associated prime of a decomposable ideal.

1. Let $M_1, M_2 \subseteq_R M$ be R-modules and $M' = M_1 \cap M_2$. Show that
$$\operatorname{Ass}_R M/M' \subseteq \operatorname{Ass}_R M/M_1 \cup \operatorname{Ass}_R M/M_2.$$
2. Determine the associated primes of a finitely generated Abelian group (viewed as a $\mathbb{Z}$-module) using the structure theorem of finitely generated Abelian groups.

Extra Problem 6.

Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in the field $\mathbb{K}$. Recall that an ideal is *monomial* if it has a generating set which consist of monomials

$\mathbf{x}^{\mathbf{d}} := x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$, with $\mathbf{d} \in \mathbb{N}^n$. You may use the following facts: a polynomial $f = \sum c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$, with each $c_{\mathbf{a}} \neq 0$, belongs to the monomial ideal I if and only if each of the monomials $\mathbf{x}^{\mathbf{a}}$ (i.e. those appearing with nonzero coefficient in f) belongs to I. This implies that each monomial ideal has a unique minimal set of monomial generators.

1. Show that a monomial ideal of the form $(x_{i_1}^{d_1}, \ldots, x_{i_r}^{d_r})$ for some $i_1, \ldots, i_r \in \{1, \ldots, n\}$ and $d_i \in \mathbb{N}_{>0}$ is $(x_{i_1}, \ldots, x_{i_r})$-primary. We call such an ideal a **basic primary ideal**.
2. Compute a minimal generating set of $(x_1, x_2^2, x_3^3) \cap (x_4, x_3^2, x_2^3)$.
3. Describe the intersection of two arbitrary **basic** primary ideals.
4. Find a primary decomposition of $(x_1 x_2^2 x_3^3, x_2^3 x_3^2 x_4)$ into basic primary ideals, as well as a minimal primary decomposition.