Freie Universität Berlin
Institut für Mathematik
Stand: 11. Dezember 2024

WiSe24/25
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Algebra I – Homework 9

Deadline: **20:00 on Wednesday 18.12.2024.** (Uploads are still possible until Friday 20.12 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H9.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Is being free a local property? What about being finitely generated?

Problem 2. **2 points**

Let U be a multiplicatively closed subset of a ring R, and let M be a finitely generated R-module.

1. Prove that $U^{-1}M = 0$ if and only if there exists $u \in U$ such that $uM = 0$.
2. Assume that $U = 1 + I = \{1 + a \ : \ a \in I\}$ for some ideal $I \subseteq R$. Show that $U^{-1}I$ is contained in the Jacobson radical of $U^{-1}R$.
3. Use Nakayama's Lemma and the first to questions to prove that if $M = IM$, then there exists $x \equiv 1 \bmod I$ such that $xM = 0$.

Reading assignment 3.

Read Propositions 3.26 and 3.28, including their proofs and corollary, in the lecture notes. Alternatively, you can find them in Atyiah-Macdonald on page 43.

Total: 4 points

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 4.

Let R be a ring and M an R-module. The **support** of M is defined as the following set:

$$\mathrm{Supp}(M) := \{\mathfrak{p} \in \mathrm{Spec}(R) \ : \ M_{\mathfrak{p}} \neq 0\}.$$

Show that:

1. If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of R-modules, then
$$\mathrm{Supp}(M_2) = \mathrm{Supp}(M_1) \cup \mathrm{Supp}(M_3).$$
2. If $M = \sum_i M_i$, then $\mathrm{Supp}(M) = \bigcup_i \mathrm{Supp}(M_i)$.
3. If M is finitely generated, then $\mathrm{Supp}(M) = V(\mathrm{Ann}_R(M))$.[†]
4. If M_1 and M_2 are finitely generated, then $\mathrm{Supp}(M_1 \otimes_R M_2) = \mathrm{Supp}(M_1) \cap \mathrm{Supp}(M_2)$.

Extra Problem 5.

Let M, N be two R-modules, $\varphi : M \longrightarrow N$ an R-linear map, and $\mathfrak{m} \in \mathrm{MaxSpec}(R)$. Is it true in general, that $\mathrm{Ker}(\varphi)_{\mathfrak{m}} = \mathrm{Ker}(\varphi_{\mathfrak{m}})$ and $\mathrm{Coker}(\varphi)_{\mathfrak{m}} = \mathrm{Coker}(\varphi_{\mathfrak{m}})$?

Extra Problem 6.

Let M be and R-module and $I \subseteq R$ an ideal. Prove that, if $M_{\mathfrak{m}} = 0$ for all maximal ideals $I \subseteq \mathfrak{m}$, then $M = IM$.

Extra Problem 7.

Let $R = \mathbb{K}[x]_{(x)}$ be the polynomial ring over a field $\mathbb{K}$ in one variable x, localized at the prime ideal (x). Find an R-module M that is not finitely generated, but such that M/xM is finitely generated.

Extra Problem 8.

Let $\varphi : R \longrightarrow S$ be a ring homomorphism, and $U \subseteq R$ be a multiplicatively closed set. Then the set $V := \varphi(U) \subseteq S$ is also multiplicatively closed (easy check). Show that $U^{-1}S$ and $V^{-1}S$ are isomorphic as $U^{-1}R$-modules.

[†] Recall that for an ideal $I \subset R$, $V(I) = \{\mathfrak{p} \in \mathrm{Spec}\, R \ : \ I \subseteq \mathfrak{p}\}$.