Freie Universität Berlin
Institut für Mathematik
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WiSe24/25
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Algebra I – Homework 8

Deadline: **20:00 on Wednesday 11.12.2024.** (Uploads are still possible until Friday 13.12 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H8.pdf`

Full written proofs are required in support of your answers.

Problem 1. **2 points**

Remark: For any R-module N, there exists some set $\mathcal{I}$ and an R-module K such that we have the short exact sequence: $0 \longrightarrow K \longrightarrow R^{\oplus \mathcal{I}} \xrightarrow{\alpha} N \longrightarrow 0$. This is works like this: take $\{n_i \mid i \in \mathcal{I}\}$ to be a set of generators of N, and K the kernel of the map $\alpha : R^{\oplus \mathcal{I}} \longrightarrow N$ with $\alpha(e_i) = n_i$. Use this fact in the following problem.

Let $0 \longrightarrow M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \longrightarrow 0$ be a short exact sequence of R-modules with M_3 flat.

1. Show that the sequence $0 \longrightarrow M_1 \otimes_R N \longrightarrow M_2 \otimes_R N \longrightarrow M_3 \otimes_R N \longrightarrow 0$ is exact for any R-module N.[¶]
2. Show that M_1 is flat if and only if M_2 is flat. (You may use the previous question without proving it.)

Problem 2. **2 points**

Let I be an ideal of R an $U := 1 + I = \{1 + a \ : \ a \in I\}$. Show that $U^{-1}I = \{\frac{a}{s} \ : \ a \in I, s \in U\}$ is an ideal of $U^{-1}R$ and that it is contained in the Jacobson radical of $U^{-1}R$.

Total: 4 points

[¶]Hint: Tensor the sequence of M_is with the one above and use the snake lemma.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Let R be a ring and $U \subseteq R$ be a multiplicatively closed subset. Prove or disprove the following statement:

If $f : R \longrightarrow U^{-1}R$ is a ring homomorphism, then every element in $f(U)$ is invertible.

Extra Problem 4.

For a ring R and an element a in R, write $R_a := U^{-1}R$ for $U = \{a^i \;:\; i \in \mathbb{N}\}$. For a prime ideal $\mathfrak{p} \subseteq R$ write $R_\mathfrak{p} := U^{-1}R$ when $U^{-1} = R \setminus \mathfrak{p}$.

1. Determine the localisations $(\mathbb{Z}/6\mathbb{Z})_2$, $(\mathbb{Z}/6\mathbb{Z})_3$, $(\mathbb{Z}/6\mathbb{Z})_{(2)}$, $(\mathbb{Z}/6\mathbb{Z})_{(3)}$. Are the respective canonical localisation maps $f : \mathbb{Z}/6\mathbb{Z} \longrightarrow (\mathbb{Z}/6\mathbb{Z})_\bullet$ injective or surjective?
2. A ring homomorphism $f : R \longrightarrow S$ is **flat**, if it turns S into a flat R-module. Give two different proofs that the ring homomorphism $\mathbb{Z}/6\mathbb{Z} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}/2\mathbb{Z}$ is flat.
3. Let $U = \{1, 3\} \subseteq \mathbb{Z}/6\mathbb{Z}$. Check for which ideals I of $\mathbb{Z}/6\mathbb{Z}$, the localization map $f : \mathbb{Z}/6\mathbb{Z} \longrightarrow U^{-1}(\mathbb{Z}/6\mathbb{Z})$ satisfies $I = f^{-1}(U^{-1}I)$.

Extra Problem 5.

Let S be an R-algebra with structural morphism $\varphi : R \longrightarrow S$. The restriction and the extension of scalars defined in the lecture give two functors: $\mathcal{E} : \textbf{R-mod} \longrightarrow \textbf{S-mod}$ with $\mathcal{E}(M) := S \otimes_R M$, and $\mathcal{R} : \textbf{S-mod} \longrightarrow \textbf{R-mod}$ with $\mathcal{R}(N) =: {}^R N$, that is N viewed as an R-module via φ.

1. Prove that functors $\mathcal{E}$ and $\mathcal{R}$ are adjoint to each other. Which is left and which is right?
2. Check $\mathcal{E}$ and $\mathcal{R}$ for left/right exactness. Do you really need to check all four, or can you use the previous question?

Extra Problem 6.

We have seen that for a ring R and U a multiplicatively closed set, $U^{-1}R = 0$ is equivalent to $0 \in U$. If M is a finitely generated R-module, what is an equivalent condition for $U^{-1}M = 0$? (Formulate and prove it.)