Freie Universität Berlin
Institut für Mathematik
Stand: 27. November 2024

WiSe24/25
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Algebra I – Homework 7

Deadline: **20:00 on Wednesday 4.12.2024.** (Uploads are still possible until Friday 6.12 at 23:55)

Submission: individually, on Whiteboard as `LASTname_A1_H7.pdf`

Full written proofs are required in support of your answers.

Problem 1. — 2 points

Let I be an ideal of the ring R and M an R-module.

1. Show that $R/I \otimes_R M$ is isomorphic to M/IM.
2. When $(R, \mathfrak{m})$ is a local ring and M, N are finitely generated R-modules, prove that if $M \otimes_R N = 0$, then $M = 0$ or $N = 0$. †

Problem 2. — 2 points

Let $n, m \in \mathbb{N}_{>0}$.

1. Prove that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = 0$ if m and n are coprime. What happens when m and n are not coprime?
2. Compare $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$ with $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}/12\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$.

Total: 4 points

†Hint: Use the previous questions and Nakayama.

Extra Problems

These problems are neither to be graded nor need to be submitted. They will be discussed in the exercise session and are highly recommended for exam preparation.

Extra Problem 3.

Prove that $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) = 0$.

Extra Problem 4.

Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$. Is it also true that $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ is isomorphic to $\mathbb{R}$?

Extra Problem 5.

Describe $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, with $n \in \mathbb{N}$.

Extra Problem 6.

Let R be a ring and $I, J \subseteq R$ ideals. Find $K \subseteq R$ such that $R/I \otimes_R R/J \cong R/K$.

Extra Problem 7.

Let $\mathbb{K}$ be a field and $m, n \in \mathbb{N}$.

1. Describe $\mathrm{Hom}_{\mathbb{K}[x]}(\mathbb{K}[x]/(x^n), \mathbb{K}[x]/(x^m))$ as explicitly as possible.
2. Describe $\mathbb{K}[x]/(x^n) \otimes_{\mathbb{K}[x]} \mathbb{K}[x]/(x^m)$ as explicitly as possible.

Extra Problem 8.

For any R-module M define $M[x]$ to be the module of polynomials in the variable x with coefficients in M, that is expressions of the form $m_0 + m_1 x + \ldots m_n x^n$, with $m_i \in M$. With the natural addition of polynomials this is an Abelian group, and with the natural multiplication with polynomials from $R[x]$ it becomes an $R[x]$-module.

1. Show that $M[x] \cong R[x] \otimes_A M$.
2. Show that if $\mathfrak{p} \subseteq R$ is a prime ideal, then $\mathfrak{p}[x]$ is a prime ideal of $R[x]$.
3. Is $\mathfrak{m}[x]$ a maximal ideal of $R[x]$, if $\mathfrak{m}$ is a maximal ideal of R?